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On integrable models from pp-wave string backgrounds

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Abstract: We construct solutions of type-IIB supergravity with non-trivial Ramond-Ramond 5-form in ten dimensions by replacing the transverse flat space of pp-wave backgrounds with exact $N = (4,4)$ $c = 4$ superconformal field theory blocks. These solutions, which also include a dilaton and (in some cases) an anti-symmetric tensor field, lead to integrable models on the world-sheet in the light-cone gauge of string theory. In one instance we demonstrate explicitly the emergence of the complex sine-Gordon model, which coincides with integrable perturbations of the corresponding superconformal building blocks in the transverse space. In other cases we arrive at the supersymmetric Liouville theory or at the complex sine-Liouville model. For axionic instantons in the transverse space, as for the (semi)-wormhole geometry, we obtain an entire class of supersymmetric pp-wave backgrounds by solving the Killing spinor equations as in flat space, supplemented by the appropriate chiral projections; as such, they generalize the usual Neveu-Schwarz five-brane solution of type-IIB supergravity in the presence of a Ramond-Ramond 5-form. We also present some further examples of interacting light-cone models and we briefly discuss the role of dualities in the resulting string theory backgrounds.

Keywords: Superstrings and Heterotic Strings, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

String theory on plane-fronted wave backgrounds, such as pp-waves, has been studied extensively in the past for two main reasons. First, under certain circumstances, they provide solutions to string theory that are exact to all orders in α', and hence their properties are fully captured by the lowest order effective supergravity [1, 2]. Second, in many cases, they provide exactly solvable string models, where the gravitational background is non-trivial and yet tractable, as in the simpler case of strings propagating in flat Minkowski space [3, 4, 5]. The main advantage of these backgrounds is given by the existence of a null isometry, which is adapted to the light-cone gauge, where string theory can be quantized in relatively simple terms. Plane-fronted wave geometries can be written using Brinkman (instead of Rosen) coordinates in the form

$$ds^2 = -4dx^+dx^- - F(y;x^+)\left(dx^+\right)^2 + dy_i dy^i\,, \tag{1.1}$$

where the front factor F may depend in general on both x^+ and the transverse coordinates y^i. Thus, in the light-cone gauge, $x^+ = \tau$, one obtains a two dimensional action that involves interactions among the world-sheet bosons y^i, which are summarized by F; if the front factor depends explicitly on x^+, the corresponding potential will also be world-sheet time dependent. As a result, and also depending on which other fields are turned on in the given background, the light-cone quantization can be carried out explicitly for those solutions that admit a rather simple form for the front factor, typically quadratic in the transverse space coordinates, or even in some more complicated situations that will be encountered later.

Recently, there has been revived interest in understanding string theory on pp-wave backgrounds following a recent proposal that the Penrose limit of superstring theory on $AdS_5 \times S^5$, which is described by a maximally supersymmetric pp-wave in ten dimensions [6], selects on the gauge theory side of the SU(N) super-Yang-Mills theory (with coupling constant $g_{\rm YM}$) certain operators with large R-charge J with $g_{\rm YM}^2 N/J^2$ kept fixed [7]. This is a special limit of the more general AdS/CFT correspondence that was proposed earlier [8] and reduces gauge theory computations to a supergravity framework via the holographic principle. Subsequently, there has been a lot of activity in understanding the physical implications of the Penrose limit in more general terms, and in favor of exploring further the connection between supergravity and gauge theories (see, for instance, [9, 10]), as any gravitational solution admits a plane-wave limit by zooming along a suitably selected null geodesic of the background geometry [11]; see also [12] that cover some of the more recent extensions. This general result makes it even more attractive to study string theory on pp-waves, as these solutions provide a consistent limit of all gravitational backgrounds of string theory that are otherwise impossible to treat in exact terms within our current understanding of the whole subject. The Penrose limit is a universal limit that may also give hints on how to proceed with the quantum theory of strings on more general backgrounds. Thus, it is not surprising that this area of research has received considerable share of attention in the recent developments in string theory and gauge theories.

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One important new aspect of the latest developments in string theory is the appearance of pp-wave geometries that also carry a non-trivial Ramond-Ramond (R-R) 5-form in the case of type-IIB critical superstring models. Since the presence of an R-R form is necessary to establish the supersymmetric properties of the corresponding pp-wave solutions, as for the maximally supersymmetric plane wave solution, and they also play a prominent role in the string theory/gauge theory correspondence, it has been proposed to consider exactly solvable models with R-R fields in the light-cone gauge, and try to quantize them. In the maximally supersymmetric solution, the front factor is independent of x^+ and provides positive mass-squared terms for all bosons on the world-sheet, thus leading to a rather simple, yet quite interesting, quantization problem [13] (but see also [14] in the presence of D-branes). In more complicated cases, where the front factor describes non-linear interactions of the string model, the situation is in general very difficult to handle in exact terms unless the corresponding two dimensional field theory becomes integrable. Then, modulo some technicalities that have to do with the definition of the theory on a two dimensional cylinder rather than a plane, the quantization can be carried out rather explicitly by relying on known facts about the quantization of integrable field theories.

A complete list of gravitational backgrounds that lead to supersymmetric interacting field theories on the world-sheet has been recently worked out by Maldacena and Maoz by solving the Killing spinor equations directly on pp-wave backgrounds in the presence of a non-trivial R-R 5-form [15]. Their solutions involve only the metric, which can be flat or more generally Ricci flat in the transverse space, and a front factor that is independent of x^+, i.e., $F(y)$. It was further shown in general terms that all such supersymmetric solutions provide exact backgrounds of string theory [16]. In a related recent development, a generalization was also considered by turning on a null Neveu-Schwarz–Neveu-Schwarz (NS-NS) and/or an R-R 3-form in the target space, in the sense that these fields have non-vanishing components along the x^+ direction, as for the R-R 5-form [17]; these supersymmetric backgrounds were also shown to be exact to all orders in α' in close analogy with ordinary pp-wave solutions with null torsion. Certainly, a systematic generalization of these results becomes important when more background fields are turned on, and in many different ways, in the presence of a non-trivial self-dual R-R 5-form.

In the present work we study systematically new solutions that can be obtained by replacing (at least) four of the eight transverse directions of a pp-wave by exact conformal field theory blocks with $N = (4, 4)$ world-sheet supersymmetry. These models have the advantage of being exact to all orders in α', thus rendering the full ten dimension solution exact in string theory. Also, they include a number of exact conformal field theory backgrounds that can be formulated as Wess-Zumino-Witten models, and as such they can be studied systematically from the world-sheet point of view, which is important for the light-cone formulation of the problem; the front factor that is induced by the presence of a non-trivial R-R 5-form can then be understood as deformation of the corresponding superconformal field theories, which in many interesting cases can be integrable. Thus, to conform with the general program of the light-cone quantization of string theory, one may employ results from the integrable deformations of two dimensional conformal field theories in order to provide exactly solvable models of interacting quantum theories on the string world-sheet.

Our focus is put on models with a large amount of world-sheet supersymmetry, which in some cases can also be manifestly space-time supersymmetric in the presence of non-trivial dilaton and NS-NS anti-symmetric tensor fields. The implications of various dualities within the class of the selected models can also be studied in order to generate some new solutions. There is a special type of such models that can be constructed by embedding axionic instantons in the transverse space, which are also space-time supersymmetric. For this particular class, the transverse space metric is conformally flat and the solution of the Killing spinor equations reduces to the Maldacena-Maoz solution as in flat transverse space, after imposing the appropriate chiral projections. Then, one expects in general that more arbitrary $N = 4$ superconformal theories with torsion, which have their target space metric conformally equivalent to a hyper-Kahler metric, will also admit systematic solutions by reducing the problem to the Killing spinor equations of a purely gravitational hyper-Kahler background by conformal transformation; this generalization, however, will not be spelled out in any detail in the present work. One way to think of the solutions that we construct explicitly here is as being generalizations of the usual five-brane solutions [18] in the presence of an R-R 5-form, in effect, this induces a $(dx^+)^2$ perturbation of the metric driven by the corresponding front factor that transforms the remaining six dimensional Minkowski space of the usual five-brane solution into a (special) pp-wave form. In fact, we will be able to obtain some interesting integrable systems on the string world-sheet by putting the (semi)-wormhole geometry in the transverse directions or, for that matter, any other exact $N = (4, 4)$ background related to it by various dualities.

It is important to realize that there can be interesting examples of exactly solvable models in our general class of superconformal theories, which will not have manifest space-time supersymmetry if their $N = (4, 4)$ world-sheet supersymmetry is non-locally realized in terms of parafermion fields [19, 20]. In those cases, we will only be able to obtain some sporadic solutions by solving directly the second order equations, as it does not make much sense to look for solutions of the Killing spinor equations in their background geometries. Since all models with non-locally realized world-sheet supersymmetry can be mapped into axionic instantons, using T-duality transformations with respect to non-triholomorphic Killing vector fields, one may further inquire whether the classification of all space-time supersymmetric solutions induced by axionic instanton backgrounds in the transverse space can also provide a systematic list of solutions for their T-dual faces. Clearly, in some cases, it might be possible to generate certain new solutions by T-duality within the type-IIB theory, but in general the dual background will not permit the consistent choice of an R-R 5-form for any prescribed front factor; we will encounter some examples of when this can happen or cannot happen later. In any case, it is known that T-duality is a symmetry that relates type IIB with type-IIA theory [21], and so in general one may only use it systematically to construct new solutions with a consistent type IIA interpretation (see also [22]).

The remaining paper is organized as follows. In section 2, we review the basic elements of interacting light-cone models by considering the classification of all space-time supersymmetric solutions in the simplest case of purely gravitational backgrounds carrying a self-dual R-R 5-form. We will expose some aspects of the general construction, which are important for the more general class of solutions that will be presented later. In section 3, we study generalized gravitational R-R backgrounds for pp-wave geometries in the presence of non-trivial dilaton and NS-NS anti-symmetric tensor fields in the transverse space, but without having any components or dependence of these fields on the light-cone coordinate x^+. We also formulate our proposal to use $N = (4,4)$ superconformal building blocks as internal theories in the transverse space, and outline a short proof for the exact nature of such gravitational backgrounds in string theory. In section 4, we present a list of superconformal building blocks that admit an exact description as Wess-Zumino-Witten models, and which can be put in the transverse space to construct exact pp-wave solutions with interesting geometrical properties. Among these exact solutions, the (semi)-wormhole geometries are also included to provide generalized five-brane solutions in the presence of R-R fields. In section 5, we construct some explicit solutions of the second order field equations by making appropriate (yet consistent) choices for the front factor F and the R-R field. Here, we also interpret the resulting interacting light-cone models as integrable perturbations of the underlying exact Wess-Zumino-Witten models, and consider some special cases where T-duality can be employed as a solution generating transformation within type-IIB theory. The integrable systems that we encounter include the complex sine-Gordon model, the supersymmetric Liouville theory and the complex sine-Liouville model in two dimensions. In section 6, we study systematically the case where axionic instantons are placed in the transverse space and show that the classification of space-time supersymmetric solutions reduces to the simpler problem of supersymmetric pp-wave backgrounds with flat transverse space by a conformal rotation and appropriate chiral projections. Thus, new examples of interacting light-cone models can be constructed with support on axionic instanton spaces. In section 7, we briefly discuss the possibility to place double axionic instanton solutions in the full eight dimensional transverse space and present some simple solutions. We also briefly discuss some general aspects of the type-IIA–IIB duality for pp-wave backgrounds in view of future applications of the current work. Finally, in section 8, we present our conclusions and indicate some general future directions of research.

2. Interacting light-cone string models

In this section we review the construction of solutions of type-IIB supergravity in ten dimensions with non-trivial Ramond-Ramond 5-form in spaces with a null Killing isometry. More precisely, following [15], we consider the ansatz for the metric g and the self-dual R-R 5-form $\mathcal{F}$,

$$\begin{aligned} ds^2 &= -4dx^+dx^- - F(y)\left(dx^+\right)^2 + h_{ij}(y)dy^i dy^j \,, \\ \mathcal{F} &= dx^+ \wedge \varphi(y)\,, \end{aligned} \tag{2.1}$$

whereas all other fields of the theory are set equal to zero. Later, we will generalize the construction to spaces with non-trivial dilaton and anti-symmetric NS-NS tensor fields. Here, we assume that the front fator F and the metric h of the transverse space depend only on the transverse coordinates y^i with $i = 2, 3, \ldots, 9$. Also, self-duality of $\mathcal{F}$ in ten dimensions implies that $\varphi(y)$, which is also taken to depend on the transverse coordinates, is an anti-self-dual 4-form in the transverse space.

It is known that the field equations in this purely gravitational target space simply read as follows,

$$d\varphi = 0\,, \qquad R_{ij}[h] = 0\,, \qquad \nabla^2 F = \frac{1}{2 \cdot 4!} \varphi_{ijkl} \varphi^{ijkl}\,, \tag{2.2}$$

where ∇^2 is the laplacian in the transverse space and the normalization of φ is fixed accordingly. Therefore, in the absence of any other fields, it follows that the transverse space has to be Ricci flat and the anti-self-dual 4-form φ has to be closed, and hence co-closed. Then, the problem is to find solutions of the second order equation that relates the norm of φ with the front factor F for this general class of pp-wave type backgrounds[1]. These solutions will in turn give rise to interacting string models in the light-cone formulation, as the front factor F can be a highly complicated function of y by making appropriate choices of the R-R form.

2.1 Anti-self-dual closed 4-forms

We present first some basic facts about the classification of anti-self-dual closed 4-forms in eight dimensions, which will be of general value in the sequel, as well as in all remaining sections. For this, it is useful to consider transverse spaces that admit complex structure and denote the complex coordinates by u, w, v, z plus their complex conjugates. Then, in complex notation, we have in general the occurrence of $(4, 0)$ or $(0, 4)$ or $(1, 3)$ or $(3, 1)$ or $(2, 2)$ forms. In order to examine which of them can be anti-self-dual, we introduce the fully anti-symmetric tensor in eight dimensions with $\epsilon_{uwvz}{}^{uwvz} = 1$, which is fully covariantized and its indices can be raised or lowered by the Kahler metric h of the ambient space. Since the Poincare dual of the $(4, 0)$ form equals to itself, the condition of anti-self-duality makes it vanish and there can be no anti-self-dual forms of this kind; likewise, there can be no non-vanishing anti-self-dual $(0, 4)$ form. As for the remaining forms, one has to consider appropriate combinations that select the anti-self-dual components, since any form can always be written as a sum of self-dual and anti-self-dual terms.

To motivate the presentation, it is useful to compared this situation with the anti-self-dual equations for Yang-Mills fields in four dimensions, as it was originally done by Yang in his R-gauge formulation of the problem in flat euclidean space [23]. Introducing complex coordinates u and w and their complex conjugates, one finds that the $(2, 0)$ and $(0, 2)$ forms are self-dual, i.e., ${}^\star F_{uw} = F_{uw}$, ${}^\star F_{\bar{u}\bar{w}} = F_{\bar{u}\bar{w}}$; therefore, imposing anti-self-duality on the field strength makes these components vanish. On the other hand, the other components of the field strength, which are represented by $(1, 1)$ forms in complex notation, obey ${}^\star F_{u\bar{u}} = F_{w\bar{w}}$ and so the combination $F_{u\bar{u}} - F_{w\bar{w}}$ is an anti-self-dual form. Then, in this context, the anti-self-dual Yang-Mills equations read $F_{u\bar{u}} + F_{w\bar{w}} = 0$, i.e., the complementary self-dual $(1, 1)$ form vanishes. As for the remaining $(1, 1)$ form $F_{u\bar{w}}$ and its

[1]We use the terminology of pp-waves rather loosely, even if the transverse space is not flat, i.e., as for the usual plane geometry.

complex conjugate, $F_{\bar{u}w}$, it can be easily seen that they are self-dual and hence vanish by imposing anti-self-duality once more. It turns out in Yang-Mills theory that setting all self-dual forms equal to zero, which is equivalent to considering anti-instanton configurations, is also consistent with the second order classical field equations.

In the present higher dimensional setting, with a curved metric h in general, the classification is made easy by introducing the short-hand notation, as in [15],

$$\varphi_{mn} = \frac{1}{6}\varphi_{m\bar{l}\bar{p}\bar{q}}\epsilon^{\bar{l}\bar{p}\bar{q}\bar{n}}h_{n\bar{n}}\,, \qquad \varphi_{m\bar{n}} = \frac{1}{2}\varphi_{m\bar{n}p\bar{p}}h^{p\bar{p}} \tag{2.3}$$

using the corresponding $(1,3)$ and $(2,2)$ forms; the $(2,2)$ forms $\varphi_{m\bar{n}p\bar{q}}$ with all m,n,p,q different from each other are all self-dual, and hence they have been omitted from the short-hand notation above. First, as far as the anti-self-duality of $(1,3)$ forms is concerned, it simply translates to having symmetric matrices φ_{mn}, thus determining the ten $(= 16 - 6)$ different linear combinations of them, whereas the anti-symmetric part of φ_{mn}, which consists of six self-dual forms, is dismissed. Likewise, one may consider ten different anti-self-dual $(3,1)$ forms by simply taking the complex conjugate of the anti-self-dual $(1,3)$ forms that have just been discussed. Then, taking the 4-form φ as a sum of the anti-self-dual $(1,3)$ forms plus their complex conjugates, yields a real expression for the R-R 5-form $\mathcal{F}$. Next, as far as the $(2,2)$ forms are concerned, it turns out that the condition of anti-self-duality simply translates to having only those linear combinations of the $\varphi_{m\bar{n}p\bar{p}}$ that render the matrix $\varphi_{m\bar{n}}$ traceless; thus, there can be fifteen $(= 16 - 1)$ independent components of this kind. Also, since the R-R 5-form $\mathcal{F}$ has to be real, the matrix that summarizes $\varphi_{m\bar{n}}$ has to be hermitean.

Finally, the closure of the 4-form φ introduces additional dynamical constraints on the components of the corresponding anti-self-dual forms that have to be taken into account in the construction of consistent type-IIB supergravity backgrounds with non-trivial R-R 5-form fields. Actually, for 5-forms $\mathcal{F} = dC^{(4)}$ that are derived from an R-R 4-form potential $C^{(4)}$, as in all cases that we considered here, the condition of self-duality is sufficient to insure the closure and co-closure of $\mathcal{F}$.

2.2 Maldacena-Maoz supersymmetric solutions

Following the recent developments in the subject, we present a brief account of the classification of all supersymmetric solutions found in the presence of a non-trivial R-R 5-form, provided that only the metric field is taken into account. In this case, one has to solve the Killing spinor equations in ten dimensions,

$$D_\mu \epsilon \equiv \left(\nabla_\mu + \frac{i}{16}\not{\mathcal{F}}\,\Gamma_\mu\right)\epsilon = 0\,, \tag{2.4}$$

where the covariant derivative includes the contribution from the spin connection, as usual, and Γ_μ are the standard Dirac matrices. We also use the normalization $\not{\mathcal{F}} = \Gamma^{ijklr}\mathcal{F}_{ijklr}/5!$, and furthermore note that the second order equations are inert to the change of sign of the R-R field $\mathcal{F} \to -\mathcal{F}$. The equations above are required by space-time supersymmetry of the bosonic class of pp-wave string backgrounds, as they can be used to set the variation of the gravitino fields consistently equal to zero. Supersymmetric conditions reduce the second order equation for the front factor F into a simpler set of first order equations and provide solutions for F and $\mathcal{F}$ whenever Killing spinors exist. The number of Killing spinors depends on the amount of space-time supersymmetry, and as a result the structure of the solutions becomes richer as the amount of supersymmetry increases. Maldacena and Maoz were able to determine the Killing spinors explicitly and characterize completely the allowed form of the functions F and $\mathcal{F}$ in a systematic way [15]. We summarize their results below, without giving details of their proof, in a way that depends on the amount of supersymmetry.

(i) $(2,2)$ *supersymmetry*: In this case the solutions are parametrized by a holomorphic function $\mathcal{W}$ and a real Killing potential $\mathcal{U}$, which gives rise to a Killing vector field with components $V_m = i\nabla_m\mathcal{U}$ and $V_{\bar{m}} = -i\nabla_{\bar{m}}\mathcal{U}$, so that V^m is holomorphic and $V^{\bar{m}}$ is anti-holomorphic. We also have to impose the conditions, which are additionally required by supersymmetry,

$$\nabla_m V^m = 0\,, \qquad \nabla_n(V^m\nabla_m\mathcal{W}) = 0\,. \tag{2.5}$$

Then, the supergravity solution is described in complex coordinates as follows:

$$\begin{aligned} 2F &= h^{m\bar{n}}(\nabla_m\mathcal{W})(\nabla_{\bar{n}}\bar{\mathcal{W}}) + h_{m\bar{n}}V^mV^{\bar{n}}\,, \\ \varphi_{mn} &= \nabla_m\nabla_n\mathcal{W}\,, \qquad \varphi_{\bar{m}\bar{n}} = \nabla_{\bar{m}}\nabla_{\bar{n}}\bar{\mathcal{W}}\,, \qquad \varphi_{m\bar{n}} = \nabla_m\nabla_{\bar{n}}\mathcal{U}\,. \end{aligned} \tag{2.6}$$

All solutions in this class have $(2,2)$ supersymmetry or more.

Simpler solutions are obtained when $\mathcal{U}=0$, in which case there is no contribution to the R-R field from the anti-self-dual $(2,2)$ forms.

(ii) $(1,1)$ *supersymmetry*: in this case the solutions are parametrized by a *real harmonic* function U and the corresponding expressions assume the form

$$\begin{aligned} 2F &= h^{m\bar{n}}(\nabla_m U)(\nabla_{\bar{n}}U)\,, \\ \varphi_{mn} &= \nabla_m\nabla_n U\,, \qquad \varphi_{\bar{m}\bar{n}} = \nabla_{\bar{m}}\nabla_{\bar{n}}U\,, \qquad \varphi_{m\bar{n}} = \nabla_m\nabla_{\bar{n}}U\,. \end{aligned} \tag{2.7}$$

thus receiving contributions from both $(2,2)$ and $(1,3)$ forms and their complex conjugates.

Note that when the transverse directions are flat, and so we may choose $h_{ij} = \delta_{ij}$, the solutions with $(2,2)$ supersymmetry (or more) become simplified, even when $\mathcal{U} \neq 0$. In this case, the real Killing potential can be chosen as

$$\mathcal{U} = C_{m\bar{n}}z^m z^{\bar{n}}\,, \tag{2.8}$$

where z^m and $z^{\bar{m}}$ denote collectively all four complex coordinates and their complex conjugates in the flat transverse space, and $C_{m\bar{n}}$ is a constant hermitean matrix with zero trace. This is required in order for the corresponding holomorphic Killing vector field

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$V^n = -iC_m{}^n z^m$ to be covariantly constant, which here translates to $\partial_m V^m = -iC_m{}^m = 0$. Then, according to the general form of the supergravity solution, the anti-self-dual $(2,2)$ forms $\varphi_{m\bar{n}}$ are simply given by the constant elements

$$C_{m\bar{n}} = \varphi_{m\bar{n}} \, . \tag{2.9}$$

Hence, their contribution to the front factor F is only quadratic in the complex coordinates z, as $h_{m\bar{n}} V^m V^{\bar{n}} = C_m{}^q C_{q\bar{n}} z^m z^{\bar{n}}$. Finally, in the flat space case, there should be an additional constraint on $\mathcal{W}$ provided by the condition

$$\partial_n (C_q{}^m z^q \partial_m \mathcal{W}) = 0 \tag{2.10}$$

for all n. As for the class of solutions with $(1,1)$ supersymmetry, there is nothing special happening in the flat space limit apart from the obvious fact that covariant derivatives turn into ordinary ones.

One may also verify directly that all these are solutions of the second order equations without making any reference to supersymmetry.

The maximally supersymmetric pp-wave background of type-IIB supergravity [6] can be obtained as a special case of the general scheme that was described above by choosing

$$\mathcal{W} = \frac{1}{2}\left(u^2 + w^2 + v^2 + z^2\right) ; \qquad \mathcal{U} = 0 \tag{2.11}$$

with flat transverse space. Then, the solution is described in complex coordinates by the following functions:

$$\begin{aligned} F &= \frac{1}{2}\left(|u|^2 + |w|^2 + |v|^2 + |z|^2\right) , \\ \varphi &= du \wedge d\bar{w} \wedge d\bar{v} \wedge d\bar{z} + dw \wedge d\bar{u} \wedge d\bar{v} \wedge d\bar{z} + \\ &\quad + dv \wedge d\bar{u} \wedge d\bar{w} \wedge d\bar{z} + dz \wedge d\bar{u} \wedge d\bar{w} \wedge d\bar{v} + \text{cc.} \end{aligned} \tag{2.12}$$

This simple background gives rise to massive free bosons on the string world-sheet in the light-cone formulation of string theory, which can be easily quantized [13], and one can also incorporate the presence of D-branes upon quantization [14]. More complicated solutions with non-constant R-R fields and arbitrary front factors arise by considering more general holomorphic functions $\mathcal{W}$. We will briefly mention some interesting examples of the more general solutions shortly. In those cases, the two dimensional action that describes string propagation in the light-cone gauge, may contain non-linear interaction terms among the world-sheet bosons, which can be studied exactly when there is an integrable system at work on the string world-sheet.

2.3 World-sheet light-cone action

Gravitational backgrounds with a null Killing isometry, as in the general class of the string models we are studying here, admit a light-cone formulation as in [2] for ordinary plane-fronted wave backgrounds. The bosonic part of the two dimensional action of a string propagating in such backgrounds is

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \left(-4\partial_a X^+ \partial^a X^- - F(y)\partial_a X^+ \partial^a X^+ + h_{ij}(y)\partial_a y^i \partial^a y^j\right) , \tag{2.13}$$

where σ and τ are the world-sheet coordinates, and the world-sheet metric is chosen to be flat by imposing the conformal gauge. Note that the front factor F is only determined up to an additive constant,[2] as we can always reparametrize X^- to $X^- + cX^+$. As usual, extremizing with respect to X^-, yields the Laplace equation $\partial_a \partial^a X^+ = 0$, and so we may fix the remaining gauge invariance by choosing the time-like parameter of the world-sheet as $X^+ = 2\pi\alpha' p^+ \tau$, where the constant p^+ is the + component of the momentum density. From now on, we normalize the parameters so that the light-cone gauge simply reads $X^+ = \tau$.

We may check the consistency of the light-cone gauge for this generalized class of pp-wave backgrounds with non-trivial transverse space metric by writing the equations of motion for the fields X^- and y^i, together with the two Virasoro constraints,

$$\begin{aligned} T_{++} &\equiv -4\partial X^- - F(y) + h_{ij}(y)\partial y^i \partial y^j = 0\,, \\ T_{--} &\equiv -4\bar{\partial} X^- - F(y) + h_{ij}(y)\bar{\partial} y^i \bar{\partial} y^j = 0\,, \end{aligned} \tag{2.14}$$

where we have used derivatives with respect to the light-cone coordinates on the string world-sheet. It can be easily checked, but we spare the details which are similar to the case of transverse spaces with flat metric (see for instance [24], but also [2]), that the Virasoro constraints provide two first order equations for the field X^-, which are compatible provided that y^i satisfy their own classical equations of motion; thus, they may be used to uniquely determine X^- up to an arbitrary zero mode. Summarizing, the general class of pp-wave geometries with arbitrary metric in their transverse space can all be studied systematically by imposing the light-cone gauge and then quantizing.

The world-sheet action becomes in the light-cone gauge

$$S_{\rm lc} = \int_\Sigma \left(h_{ij}(y)\partial y^i \bar{\partial} y^j - F(y) \right) , \tag{2.15}$$

where Σ is a two dimensional cylinder due to the periodicity of the σ variable. The full action is obtained by also adding the contribution of the fermionic fields, which are omitted here. For generic (but consistent) choices of F we obtain interacting light-cone models, which generalize the simple case of having massive free bosons on the world-sheet for the maximally supersymmetric pp-wave string background [13]. In case that we also have an anti-symmetric tensor field B_{ij} (torsion) in the classical background, as in subsequent sections, the story repeats itself, but with $h_{ij}(y)$ being replaced by $h_{ij}(y) + B_{ij}(y)$ in the light-cone action.

An interesting and characteristic example of string backgrounds with R-R fields that lead to integrable theories on the string world-sheet is provided by the $(2,2)$ supersymmetric solution in flat transverse space, with

$$\mathcal{W} = \cos u\,, \qquad \mathcal{U} = 0 \tag{2.16}$$

[2] This amounts to the freedom of shifting the vacuum energy of the light-cone sigma models by an arbitrary constant.

within the Maldacena-Maoz framework. Here, the superpotential is taken to depend only on one complex coordinate, which can be parametrized as $u = \theta + i\rho$. In this case, we find that the front factor F, which provides the non-linear interactions in the light-cone gauge is given by

$$2F = |\sin u|^2 \equiv \sin^2\theta + \sinh^2\rho\,. \tag{2.17}$$

The underlying integrable system is the $N = 2$ sine-Gordon model [25], but here it is defined on a cylinder rather than a two dimensional plane. A ten dimensional solution of type-IIB supergravity can be constructed by choosing the anti-self-dual closed 4-form φ as the following $(1,3)$ form, plus its complex conjugate [15]:

$$\varphi = -(\cos u)du \wedge d\bar{w} \wedge d\bar{v} \wedge d\bar{z} + \text{cc}. \tag{2.18}$$

Another interesting example is provided by the choice

$$\mathcal{W} = u - \frac{1}{3}\alpha^2 u^3\,, \qquad \mathcal{U} = 0\,, \tag{2.19}$$

which yields a $(2,2)$ solution with quartic interactions having

$$2F = \left|1 - \alpha^2 u^2\right|^2 \tag{2.20}$$

and an anti-self-dual closed 4-form

$$\varphi = -(2\alpha^2 u)du \wedge d\bar{w} \wedge d\bar{v} \wedge d\bar{z} + \text{cc}. \tag{2.21}$$

Light-cone quantization proceeds by employing the solvability of the underlying integrable systems, thus determining in principle the allowed string spectrum. It should be realized, nevertheless, that many results about integrable systems have to be generalized accordingly, in order to take into account the periodicity of the world-sheet coordinate σ. Thus, apart from the ordinary periodic solutions there can be other topological sectors in the model with non-trivial winding or twisting, depending on discrete symmetries; for example, for the sine-Gordon model, we have invariance of the equations under $u \to u + 2\pi$ and $u \to -u$, whereas for the quartic potential we only have invariance under $u \to -u$. Imposing strict periodicity destabilizes the soliton solutions, but stability can be regained by winding and/or twisting (see, for instance, [26]). In any case, the quantization of integrable systems on the cylinder is a less studied problem, due to the technicalities associated with periodicity, and they have to be shorted out before the spectrum of the corresponding string models can be computed in closed form. We do not really have anything more concrete to say about these particular issues in the present work, apart from a few extra clarifying remarks in the conclusions of the paper.

3. Generalized gravitational string backgrounds

We begin our study of more general gravitational backgrounds of type-IIB supergravity with a null Killing vector field and a R-R 5-form by turning on a non-trivial dilaton as well as a NS-NS anti-symmetric tensor field. As we will see later, this provides an interesting

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generalization of the pure gravitational backgrounds with R-R fields, which is still tractable and yields a variety of integrable systems in the light-cone gauge of string theory. We will first describe an ansatz that leads to consistent reduction of the β-function equations for the background fields, and then formulate a proposal for constructing some new exact solutions. The class of backgrounds that will be used in the transverse space of the ten dimensional theory will exhibit $N = (4,4)$ world-sheet supersymmetry, which will in turn imply that the proposed solutions are actually exact to all orders in α'; a short proof of this fact will also be given in this section following some standard arguments about the exactness of pp-wave geometries. Thus, although our starting point is the lowest order effective theory of type-IIB supergravity, the results we describe are also exact in string theory.

3.1 Type IIB string background equations

We consider the basic equations of type-IIB supergravity in ten dimensions in the presence of a metric g, a dilaton Φ, an anti-symmetric NS-NS tensor field that is represented by a 2-form B with field stregth $H = dB$, and an R-R 5-form $\mathcal{F}$. In general, type-IIB supergravity also has an R-R scalar $C^{(0)}$ and an R-R 2-form $C^{(2)}$ that accompany the NS-NS fields Φ and B, in which case the R-R 5-form $\mathcal{F}$ is given by the expression

$$\mathcal{F} = \partial C^{(4)} + B\partial C^{(2)} - C^{(2)}\partial B \,, \tag{3.1}$$

where $C^{(4)}$ is the corresponding R-R 4-form potential appropriately normalized. In all cases, $\mathcal{F}$ is a self-dual 5-form in ten dimensions, i.e., ${}^{\star}\mathcal{F} = \mathcal{F}$; self-duality does not admit a natural derivation from a covariant action principle that can select only the self-dual piece of $\mathcal{F}$ to the physical propagating degrees of freedom. Thus, this condition has to be implemented as an on-shell constraint when an action principle is used to describe all other equations of motion.

Setting $C^{(0)} = 0 = C^{(2)}$ for the remaining R-R fields, the supergravity equations can be derived from the effective action [24, 27] (but see also [22]),

$$S_{\rm eff} = \int d^{10}x\sqrt{-g}e^{-2\Phi}\left(R + 4(\nabla_\mu\Phi)(\nabla^\mu\Phi) - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{e^{2\Phi}}{4\cdot 5!}\mathcal{F}_{\mu\nu\rho\sigma\tau}\mathcal{F}^{\mu\nu\rho\sigma\tau}\right), \tag{3.2}$$

which is written in the σ-model frame, and they assume the form

$$\begin{aligned} R_{\mu\nu} &= -2\nabla_\mu\nabla_\nu\Phi + \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \frac{e^{2\Phi}}{4\cdot 4!}\left(\mathcal{F}_{\mu\kappa\lambda\rho\sigma}\mathcal{F}_\nu{}^{\kappa\lambda\rho\sigma} - \frac{1}{10}g_{\mu\nu}\mathcal{F}_{\kappa\lambda\rho\sigma\tau}\mathcal{F}^{\kappa\lambda\rho\sigma\tau}\right), \\ 0 &= \nabla_\mu\nabla^\mu\Phi - 2(\nabla_\mu\Phi)(\nabla^\mu\Phi) + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\,, \\ 0 &= \nabla_\mu\left(e^{-2\Phi}H^{\mu\nu\rho}\right), \\ 0 &= \nabla_\mu\mathcal{F}^{\mu\nu\kappa\lambda\rho}\,, \end{aligned} \tag{3.3}$$

where the sef-duality of $\mathcal{F}$ is also imposed as an additional constraint. Here, Greek indices take the values $0,1,2,\ldots,9$. The string equations arise as conditions for the vanishing of the β-functions to lowest order in α' and provide gravitational backgrounds that are consistent with the requirement of conformal invariance on the string world-sheet. We note for notational purposes that the corresponding equations in the Einstein frame can be obtained by passing to the ten dimensional metric $g'_{\mu\nu} = \exp(-\Phi/2)g_{\mu\nu}$. For backgrounds that exhibit enough supersymmetry, typically backgrounds with $N = (4,4)$ supersymmetry on the world-sheet, the solutions that one obtains are also exact to all orders in α' and, hence, the present framework is sufficient for their complete description. It is also useful to recall in this context that the equation for the dilaton field, which in the Einstein frame of the ten-dimensional theory simply reads $e^{\Phi}\nabla^2\Phi + H^2/12 = 0$, is derived under the assumption that there is no potential term for Φ, as we are considering critical string theory in $D = 10$ dimensions.

Next, we simplify the string background equations by restricting attention to metrics with a null Killing isometry,

$$ds^2 = -4dx^+dx^- - F(y)(dx^+)^2 + h_{ij}(y)dy^i dy^j \, ; \qquad i,j = 2,3,\ldots,9 \, , \tag{3.4}$$

where the front factor F is taken to depend only on the coordinates y that parametrize the transverse space with metric $h_{ij}(y)$, as before. We also assume that the dilaton Φ depends only on y, while the 2-form $B = B(y)$ is assumed to live entirely in the transverse space having zero components in the $x^{\pm}$ directions. As for the R-R 5-form, we also assume, as before, that

$$\mathcal{F} = dx^+ \wedge \varphi(y) \, , \tag{3.5}$$

where $\varphi(y)$ is an anti-self-dual closed 4-form in the transverse eight-dimensional space, i.e.,

$$\varphi(y) = -{}^\star\varphi(y) \, , \qquad d\varphi(y) = 0 \, , \tag{3.6}$$

in order to insure the self-duality of $\mathcal{F}$ in ten dimensions together with its conservation law. Taking the $(++)$-component of the equation for the Ricci tensor, we obtain

$$\nabla_i \left(e^{-2\Phi}\nabla^i F\right) = \frac{1}{2\cdot 4!}\varphi_{ijkl}\varphi^{ijkl} \, , \tag{3.7}$$

where all Latin indices and covariant derivatives are refering to the transverse space. In order to simplify the expressions in the sequel, we will use the the short-hand notation for the norm of the 4-form

$$|\varphi|^2 = \frac{1}{4!}\varphi_{ijkl}\varphi^{ijkl} \, . \tag{3.8}$$

The remaining equations we obtain in the transverse space are

$$\begin{aligned} R_{ij}[h] &= -2\nabla_i\nabla_j\Phi + \frac{1}{4}H_{ikl}H_j{}^{kl} \, , \\ 0 &= \nabla_i\nabla^i\Phi - 2(\nabla_i\Phi)(\nabla^i\Phi) + \frac{1}{12}H_{ijk}H^{ijk} \, , \\ 0 &= \nabla_i\left(e^{-2\Phi}H^{ijk}\right) \, , \end{aligned} \tag{3.9}$$

and therefore we have non-vanishing curvature in the presence of non-trivial dilaton and anti-symmetric tensor fields, namely

$$R[g] = R[h] = -2\nabla^2\Phi + \frac{1}{4}H^2 \, . \tag{3.10}$$

We observe that if we have a metric $h(y)$, a dilaton field $\Phi(y)$ and a 2-form $B(y)$ that solve the string background equations in $D-2=8$ dimensions with local coordinates y, then a solution of the type-IIB supergravity can also be obtained in $D=10$ dimensions with a null Killing isometry as given by the ansatz (3.4). The front factor $F(y)$ and the anti-self-dual closed 4-form $\varphi(y)$ that characterize the resulting ten-dimensional background are simply related to each other by equation (3.7), but they remain arbitrary otherwise. Here, we assume (at least to first order in α') that the central charge deficit of the $(D-2)$-dimensional theory is zero, $\delta c_{D-2}=0$, so that the total central charge in D dimensions is critical, i.e., $c=2+8=10$. Moreover, taking into account some special circumstances that are described below, it can be shown that if the $(D-2)$-background is exact to all orders in α', the D-dimensional background will also be exact to all orders.

At this point it is useful to recall briefly the history of the subject concerning the exactness of various classes of pp-wave solutions. When all other fields but the metric are set equal to zero, we have the field equation $\nabla^2 F(y)=0$ in the transverse space. For $h_{ij}=\delta_{ij}$ and quadratic front factor $F(y)=A_{ij}y^iy^j$, we obtain the traceless condition for the symmetric matrix A, $\mathrm{Tr}\,A=0$, which yields pp-waves with constant polarization as solutions of superstring theory; these solutions are known to be exact string backgrounds, as all higher order terms in the string equations of motion are automatically zero [1] (see also [2] for a class of generalized pp-wave backgrounds, but without R-R fields). Relaxing the traceless condition on the matrix A for quadratic front factors, amounts to turning on a non-trivial R-R 5-form. Indeed, the simplest such background with zero dilaton and anti-symmetric tensor fields, is given by the maximally supersymmetric pp-wave solution [6],

$$\begin{aligned} ds^2 &= -4dx^+dx^- - \left(\sum_{i=2}^{9} {y_i}^2\right)(dx^+)^2 + \delta_{ij}dy^idy^j\,, \\ \varphi &= dy^2\wedge dy^3\wedge dy^4\wedge dy^5 - dy^6\wedge dy^7\wedge dy^8\wedge dy^9\,, \end{aligned} \tag{3.11}$$

which is written here using real coordinates; equivalently, we may use here the complex coordinates

$$u=y^2+iy^6\,, \qquad w=y^3+iy^7\,, \qquad v=y^4+iy^8\,, \qquad z=y^5+iy^9 \tag{3.12}$$

to conform with the complex notation of the general supersymmetric backgrounds given earlier. As we have already seen, many other backgrounds with flat or curved space metric h_{ij}, as those that have been proposed by Maldacena and Maoz by relying on a simpler set of first order equations [15] (but see also [17]), are also supersymmetric solutions, which are exact to all orders in α'.

In our attempt to find generalized solutions in the presence of non-trivial dilaton and NS-NS anti-symmetric tensor fields, we will use the observation that the transverse space fields decouple from the rest and satisfy their own β-function equations, at least within the ansatz that has been made. Space-time supersymmetry will not be a concern at first sight, but as we will find out later, when the transverse space has manifest space-time supersymmetry, the full ten dimensional solutions of pp-wave type will also be space-time supersymmetric. Here, we focus instead on a more general class of models that exhibit a large amount of world-sheet supersymmetry, as we think that this is more fundamental for the world-sheet formulation of superstring theory in the light-cone gauge; in some cases, where all world-sheet supersymmetries can be locally realized, we will also get solutions with manifest space-time supersymmetry as bonus.

We conclude the general discussion of the string background equations by examining the possible restrictions on the components of our background fields, which are imposed by the choice of R-R fields $C^{(0)} = 0 = C^{(2)}$. In particular, if these two R-R fields are set equal to zero, the general field equations will admit a consistent truncation provided that

$$\mathcal{F}_{\mu\nu\rho\kappa\lambda} H^{\rho\kappa\lambda} = 0 \tag{3.13}$$

for all space-time indices μ, ν. These additional conditions are obtained by considering the field equation for the R-R field $C^{(2)}$, and in turn they imply within our ansatz that

$$\varphi_{ijkl} H^{jkl} = 0 \tag{3.14}$$

for all transverse space indices. Clearly, they restrict the structure of the non-vanishing components of φ when there is torsion in the transverse space of the generalized pp-wave solutions; some implications will be examined shortly in the next subsection.

3.2 Classes of exact solutions

In the following, we consider the possibility to construct new type-IIB gravitational backgrounds in ten dimensions with non-trivial dilaton and anti-symmetric tensor fields, using a suitable embedding of exact $N = (4, 4)$ superconformal field theory blocks in the transverse space, namely we consider models of the form

$$ds^2 = -4dx^+dx^- - F(y)(dx^+)^2 + [N = (4, 4),\ c = 4]_1 + [N = (4, 4),\ c = 4]_2 , \tag{3.15}$$

which is schematic way of saying that the flat transverse space is replaced by exact superconformal field theories with $(4, 4)$ supersymmetry on the world-sheet. Such four dimensional building blocks have central charge deficit equal to zero to all orders in α' and therefore, they serve our purpose provided that we also include the contribution of their dilaton fields (and in some cases their anti-symmetric tensor fields). Solutions for the front factor F and the 4-form φ will be determined later. We only note here that for the special case $F(y) = 0$, and hence $\varphi = 0$, these backgrounds were studied extensively in the past as exact and stable string solutions in both type-II and heterotic superstring theories with non-trivial dilaton and anti-symmetric tensor fields [18, 19], and in many cases the full spectrum of string excitations was derived in a modular invariant way [28].

Turning on a non-vanishing R-R 5-form amounts to switching on a front factor F in the ten dimensional metric, thus generalizing the previous class of solutions. Actually, since the perturbation of the metric driven by the front factor F depends only on the transverse coordinates y, it is natural to expect that the presence of this term in the light-cone gauge of string theory can be interpreted as a perturbation of the $N = (4, 4)$ superconformal building blocks that drives them away from criticality. In the cases that we will consider in the sequel these perturbations correspond to integrable field theories on the string world-sheet, and hence one expects that these will also provide tractable interacting models with calculable string spectrum (although this particular aspect of the problem is lying beyond the scope of the present work).

In a way, among other things, our proposal generalizes the construction of five-brane solutions in ten dimensions by turning on a non-trivial R-R 5-form, and goes further beyond it. Recall first that the standard construction of five-brane solutions relies on the concept of axionic instantons [29] in order to have manifest space-time supersymmetry. Axionic instantons are special backgrounds in four dimensions with a large amount of world-sheet supersymmetry, which provide consistent bosonic solutions of the first order supersymmetric conditions

$$\left(\gamma^{\mu}\partial_{\mu}\Phi \mp \frac{1}{12}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right)\epsilon = 0\,, \qquad \left(\partial_{\mu} + \frac{1}{4}\Omega^{\alpha\beta}_{\mp\mu}\gamma_{\alpha\beta}\right)\epsilon = 0 \tag{3.16}$$

associated to the variations of the dilatino and the gravitino fields respectively, [18]. Here, $\Omega^{\alpha\beta}_{\mp\mu}$ is the usual spin connection that has been improved by subtracting or adding the torsion term for axionic instanton or anti-instanton backgrounds. The dilatino variation can be made zero by choosing the NS-NS 3-form to be Poincare dual (or anti-dual) to the derivative of the dilaton field. Then, for conformally flat metrics, with the conformal factor being equal to $\exp(2\Phi)$, the axionic instantons also provide solutions of the gravitino equation, where half of the supersymmetries are unbroken; the other half, by standard reasoning, will be associated with fermionic zero modes bound to the solitonic five-brane. Thus, axionic instantons provide a particular class of $N = (4,4)$ models that can also be used in the present context to replace four of the transverse coordinates of a pp-wave configuration.

In those cases that there is an axionic instanton block sitting in the transverse space of the generalized pp-wave configurations, the non-vanishing components of φ cannot be arbitrary, as they have to satisfy the special algebraic conditions $\varphi_{ijkl}H^{jkl} = 0$, according to equation (3.14). Then, it is reasonable to assume that $\varphi_{ijkl} \neq 0$ if two of its transverse space indices take values in the first building block and the remaining two in the second block. Otherwise, if all indices of φ take values in the axionic instanton block, we will have $\varphi_{ijkl} = 0$ for $H \neq 0$, in which case the components of the dual form with all four indices taking values in the other block will also be zero; since this is also conversely true, we rule out the possibility to have φ with all its components living only in one four-dimensional block. Likewise, if three indices take values in the axionic instanton block, it will be consistent to set those components of φ equal to zero together with the components of the corresponding dual form having only one index taking values in the axionic instanton block. The $2+2$ splitting of the transverse space indices that we are advocating here for φ amounts to a block-diagonal structure for the matrices φ_{mn} and $\varphi_{m\bar{n}}$ that describe the 4-form φ in short-hand notation; we will make further use of this structure in section 6 while describing explicit pp-wave solutions in axionic instanton backgrounds.

There is a further generalized class of candidate backgrounds with $N = 4$ supersymmetry that include torsion, which have manifest space-time supersymmetry and can also be placed appropriately in the transverse space. They provide generalizations of the standard axionic instantons, in the sense that their target space metric is conformally equivalent to an arbitrary hyper-Kahler manifold [30],

$$h = \Omega \tilde{h} \,, \qquad \tilde{\nabla}^2 \Omega = 0 \,, \tag{3.17}$$

where the conformal factor Ω satisfies the Laplace equation with respect to the conformally equivalent hyper-Kahler metric $\tilde{h}_{ij}$. They can also be viewed as generalizations of the usual hyper-Kahler geometry [31] in the presence of anti-symmetric tensor fields. These backgrounds will not be studied here in detail, but they certainly deserve separate investigation, as they can provide quite general supersymmetric solutions in our framework, which are closely related to the Maldacena-Maoz solutions in curved transverse space by conformal transformation using Ω.

Finally, there is yet another class of $N = (4,4)$ superconformal blocks with an exact four dimensional interpretation, which arise as gauged Wess-Zumino-Witten models and they are T-dual to axionic instantons. These models will be studied separately, together with their T-dual counterparts, which are (semi)-wormholes with thin or fat throats, as they constitute the only backgrounds of this kind with an exact description as Wess-Zumino-Witten models. However, unlike axionic instantons, their T-dual faces only have a dilaton but no torsion fields, which in turn imply that the dilatino variation cannot be made zero. Thus, space-time supersymmetry cannot be seen in these case despite the fact that in all these backgrounds there is an underlying extended superconformal symmetry, which makes them good candidates for building blocks of the transverse space. We will say more about this peculiar situation towards the end of section 4 by including a few extra technical details. Yet another general class of $N = 4$ superconformal theories consists of the T-dual versions of all conformally hyper-Kahler spaces with torsion; however, we will not delve into the details of these particular models in the present work, as they will be left open for future investigation, together with their T-dual faces.

3.3 On higher order α' corrections

We now turn to the exactness of the pp-wave backgrounds in string theory. The classical equations of motion for the metric in string theory, which follow from the vanishing condition of the β-function $\beta(g_{\mu\nu})$ by imposing conformal invariance of the two dimensional theory, can be expressed in terms of sigma model perturbation theory as

$$0 = R_{\mu\nu} + \frac{1}{2}\alpha' R_{\mu\rho\sigma\lambda} R_{\nu}{}^{\rho\sigma\lambda} + \cdots \,, \tag{3.18}$$

where the dots denote derivatives and higher powers of the curvature that can in principle occur to all orders in α'. In the presence of other background fields, as the dilaton, torsion and so on, there are also contributions coming from them to $\beta(g_{\mu\nu})$ to all orders in α', and similarly there can be higher order corrections to their own β-functions, $\beta(\Phi)$, $\beta(B_{\mu\nu})$, etc, involving derivatives and higher powers. The details of the general structure that appears to higher loops are well known (see, for instance, [32, 33]), and they will not be presented here. The remarkable thing about pp-wave geometries is that without even having precise knowledge of the higher order corrections, they can be shown to vanish under some general conditions.

Simple plane wave geometries with $h_{ij} = \delta_{ij}$ and no R-R fields, are known to be exact solutions to all orders in α', as it was first shown in [1] for purely metric backgrounds and in [2] in the presence of dilaton and NS-NS anti-symmetric tensor fields. The proof of the absence of higher order corrections relies entirely on the presence of a null isometry generated by a covariantly constant Killing vector field l^μ that is orthogonal to the Riemann tensor. More recently it has been extended to prove the exactness of the supersymmetric pp-wave backgrounds found by Maldacena and Maoz, as well as in some generalizations, by employing various methods [16, 17]. In our case, the proof goes in two steps: the first also relies on the presence of a null isometry, as for the simpler class of backgrounds that have been considered in the past, and the second uses the exactness of the $N = (4, 4)$ superconformal theory blocks that have been proposed to build their transverse space. It is well known for them, like for all Wess-Zumino-Witten models, that one can choose a scheme (or else a local covariant α'-dependent field redefinition) that relates the exact string background to its leading order form [32].

Taking the first step, it follows by explicit calculation that in the general class of metrics (3.4) we have,

$$
\begin{aligned}
R_{+\mu+\nu}[g] &= \frac{1}{2}\delta_{\nu,i}\delta_{\mu,j}\nabla_i\nabla_j F\,,\\
R_{+\mu\nu i}[g] &= \frac{1}{2}\delta_{\nu,+}\delta_{\mu,j}\nabla_i\nabla_j F\,,\\
R_{-\mu\nu\lambda}[g] &= 0 = R_{\mu\nu\lambda -}[g]\,,
\end{aligned}
\tag{3.19}
$$

where the covariant derivatives are taken with respect to the transverse space metric h. We also have

$$
R_{ijkl}[g] = R_{ijkl}[h]\,, \tag{3.20}
$$

whereas all other components follow by the symmetry properties of the Riemann curvature tensor under the interchange of its indices.

Using these expressions, as well as the fact that $g^{++} = 0$ for this class of metrics, we arrive immediately at the following result,

$$
R_{\mu\rho\sigma\lambda}R_\nu{}^{\rho\sigma\lambda} = R_{\mu ijk}R_\nu{}^{ijk}\,, \tag{3.21}
$$

which in turns implies that there can be no R^2 correction terms of this type to the components of the β-functions $\beta(g_{++})$, $\beta(g_{+-})$, $\beta(g_{--})$ and $\beta(g_{i\pm})$, but only to $\beta(h_{ij})$. Thus, the corrections, if any, which they assume the general form $R_{iklm}R_j{}^{klm}$, are only restricted to the conformal field theories that live in the transverse space. Likewise, any possible higher order corrections with more Riemann tensors reduce to their transverse space part, whereas all other components vanish. Possible higher order curvature terms of the form $R_{\mu\nu}R_{\lambda\rho\sigma\tau}R^{\lambda\rho\sigma\tau}$ also vanish in all cases, as before, due to the contractions of the Riemann curvature tensors, modulo terms that can be removed by appropriate field redefinitions. Finally, expressions that may involve derivatives of the curvature, like $\nabla^\rho\nabla^\sigma R_{\mu\rho\nu\sigma}$, or higher, which seem to contribute not only to the transverse space equations but to the $\beta(g_{++})$ component, can be easily removed by field redefinitions as described in [32, 17]. Summarizing the results, we see that *all higher order curvature terms of the string equations of motion* $\beta(g_{\mu\nu})$ *can be non-vanishing only for the transverse space components* $\beta(h_{ij})$, *as all metrics of pp-wave type admit a covariantly constant null Killing vector field.*

As for the possible correction terms coming from the dilaton and the NS-NS antisymmetric tensor fields, they can be easily seen to contribute to higher orders, if at all, only in the transverse space components of the β-function equations, because they do not have any components or dependence on the remaining space-time coordinates. The R-R 5-form $\mathcal{F} = dx^+ \wedge \varphi(y)$, which also depends on the coordinates of the transverse space, can also be seen to contribute no higher order correction terms in the string equations of motion. Actually, the situation here is a bit subtle and requires careful investigation. Following [17], it can be seen that higher order correction terms that involve multiple covariant derivatives of $\mathcal{F}$ can be put equal to zero by working in a natural scheme where the 2- and 3-point terms in the effective action are not modified from their supergravity values[3]. Likewise, there can not be any terms that involve derivatives of $\mathcal{F}$ contracted with curvature terms. Finally, higher order terms in $\mathcal{F}$ consisting of multiple products are also zero due to the "null" properties of the pp-wave geometries. All these arguments can be made more elaborative and exhaustive as in [17].

In any case, having arranged for all possible higher order correction terms in α' to appear only in the β-functions of the transverse space theory, which, thus, are decoupled from the rest to all orders in α', one may further take a second step and show that for all $N = (4, 4)$ superconformal building blocks of the transverse space, these higher order corrections are actually absent. In fact, one may note independently that for these lower dimensional models there exists a renormalization group scheme that makes them exact. Fortunately, this procedure, which is well known but a bit technical to carry out in detail, has already been used for the proposed class of four dimensional string backgrounds with dilaton and torsion in [32], where more details can be found. Hence, the pp-wave geometries we are considering here are not only solutions of the lowest order effective theory, but they also provide a large class of exact string backgrounds.

In conclusion, the generalizations we are considering here can be very rich in producing a wide range of non-trivial gravitational pp-wave backgrounds with R-R fields, but all of them share the same essential stringy properties with the simplest plane wave solution, which is exact to all orders in α', as in [1].

4. Exact $N = (4, 4)$ superconformal building blocks

Having presented the main idea behind our construction, we may now proceed to explicit calculations by listing some simple examples of $N = (4, 4)$ superconformal field theories, which all have central charge $c = 4$ to all orders in α'. We first recall that the Wess-

[3]Unlike the bosonic string theory, the on-shell superstring amplitudes for massless modes do not contain α' corrections, i.e., the supergravity amplitudes are exact, [24].

Zumino-Witten models

$$\mathrm{SU}(2)_k\,, \qquad \mathrm{SU}(2)_k/\,\mathrm{U}(1)\,, \qquad \mathrm{SL}(2)_k/\,\mathrm{U}(1) \tag{4.1}$$

are superconformal theories with $(2,2)$ supersymmetry on the world-sheet and central charge $c = 3k/(k+2)$, $2(k-1)/(k+2)$ and $2(k+1)/(k-2)$ respectively; we may also append the two-dimensional flat space F_2 in the above list of models, with central charge $c = 2$. Then, the following four models, which are obtained by suitable tensor products,

$$\begin{aligned} F_4 &= F_2 \times F_2\,, & \Delta_k &= \left(\frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)}\right) \times \left(\frac{\mathrm{SL}(2)_{k+4}}{\mathrm{U}(1)}\right), \\ C_k &= \left(\frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)}\right) \times \mathrm{U}(1) \times \mathrm{U}(1)_Q\,, & W_k &= \mathrm{SU}(2)_k \times \mathrm{U}(1)_Q \end{aligned} \tag{4.2}$$

all have enhanced supersymmetry to $N = (4,4)$ and central charge $c = 4$ independent of k,[4] provided that the background charge of the $\mathrm{U}(1)_Q$ factor is taken to be $Q = \sqrt{2/(k+2)}$ (see, for instance, [19] for a detailed exposition from where we also borrow the notation). The flat space F_4 will not be discussed, as it is a trivial example of a superconformal field theory; if the transverse space of the proposed ten dimensional backgrounds (3.15) is built from two F_4 superconformal blocks, we will obtain the class of the usual pp-wave backgrounds that have already been discussed in the literature [15].

Hence, we will restrict attention to models where at least one of the $N = (4,4)$ superconformal building blocks is any one from the list of non-trivial gravitational backgrounds appearing above, namely Δ_k, C_k or W_k. We will also extend our presentation to include yet another exact model, the so called fat throat model, from which all other spaces (including F_4) can be obtained by dualities and/or appropriate contractions in its parameter space. In the following, we provide a summary of their target space description using complex coordinates or equivalently using real coordinates.

(i) Δ_k: This model describes semi-classically the geometry of a two-dimensional bell times a two dimensional cigar (the euclidean two-dimensional black-hole) and has non-trivial metric and dilaton fields in four dimensions. In this case, the *negative* curvature of the compact coset cancels the *positive* curvature of the non-compact coset so that the resulting central charge deficit is $\delta c = 0$ to all orders in $1/k$. In particular, we have

$$\begin{aligned} \frac{1}{k}ds^2 &= \frac{du d\bar{u}}{1-u\bar{u}} + \frac{dw d\bar{w}}{w\bar{w}+1}\,, \\ -2\Phi &= \log(1-u\bar{u}) + \log(w\bar{w}+1) \end{aligned} \tag{4.3}$$

provided that $|u|^2 \leq 1$. In this model there is no anti-symmetric tensor field. Parametrizing the complex coordinates as

$$u = \sin\theta e^{i\phi}\,, \qquad w = \sinh\rho e^{i\chi}\,, \tag{4.4}$$

[4] Recall that $\alpha' \sim 1/k$ and hence the large level k limit is the semi-classical limit that corresponds to supergravity backgrounds to lowest order in α'. Higher curvature terms typically arise to higher orders in α', but for the $N = (4,4)$ superconformal models these terms are vanishing by supersymmetry.

we obtain the following description of the solution in real coordinates,

$$\begin{aligned}
\frac{1}{k}ds^2 &= \left(d\theta^2 + \tan^2\theta d\phi^2\right) + \left(d\rho^2 + \tanh^2\rho d\chi^2\right), \\
-2\Phi &= \log(\cos^2\theta) + \log(\cosh^2\rho). \qquad (4.5)
\end{aligned}$$

The semi-classical geometry of the SL(2)/ U(1) coset model was first studied in the literature in order to interpret solutions of two dimensional string theory as black holes [34]; the lagrangian formulation of the SU(2)/ U(1) coset was also studied along similar lines (see, for instance, [35]).

There is a subtle point that is worth emphasizing for later use, namely that the cigar geometry of the SL(2)/ U(1) factor corresponds to the axial gauging of U(1). The vector gauging is simply related to it by a T-duality transformation with respect to the Killing vector ∂_χ and turns $\tanh\rho$ to $\coth\rho$ in the metric and $\cosh\rho$ to $\sinh\rho$ in the dilaton [35]. Thus, we obtain a second version of the Δ_k model, which is given by

$$\begin{aligned}
\frac{1}{k}\tilde{ds}^2 &= \frac{dud\bar{u}}{1-u\bar{u}} + \frac{dwd\bar{w}}{w\bar{w}-1}, \\
-2\tilde{\Phi} &= \log(1-u\bar{u}) + \log(w\bar{w}-1) \qquad (4.6)
\end{aligned}$$

and describes the geometry of a two dimensional bell times a two dimensional trumpet with infinite curvature at the boundaries of the allowed range of values $|u|^2 \leq 1$, $|w|^2 \geq 1$. Note in this case that the appropriate parametrization of w changes to $w = \cosh\rho\exp(i\chi)$. On the other hand, T-duality has no effect on the topology of the SU(2)/ U(1) coset, as the bell interpretation remains unchanged under $\theta \to \pi/2 - \theta$ that summarizes the action of the corresponding T-duality on the SU(2)/ U(1) factor with respect to the isometry generated by the Killing vector field ∂_ϕ.

(ii) C_k: This model describes semi-classically the geometry of a two-dimensional bell times a two-dimensional cylinder with metric and dilaton fields given by

$$\begin{aligned}
\frac{1}{k}ds^2 &= \frac{dud\bar{u}}{1-u\bar{u}} + dwd\bar{w}, \\
-2\Phi &= \log(1-u\bar{u}) + w + \bar{w}, \qquad (4.7)
\end{aligned}$$

provided again that $|u|^2 \leq 1$. In this model there is also no anti-symmetric tensor field. Parametrizing the complex coordinates as

$$u = \cos\theta e^{i\phi}, \qquad w = \rho + i\chi, \qquad (4.8)$$

we obtain the following description of the solution in real coordinates,

$$\begin{aligned}
\frac{1}{k}ds^2 &= d\rho^2 + d\chi^2 + d\theta^2 + \cot^2\theta\phi^2, \\
-2\Phi &= 2\rho + \log(\sin^2\theta). \qquad (4.9)
\end{aligned}$$

We observe a linear dependence of the dilaton field on ρ, which is characteristic of the Liouville-like boson $\mathrm{U}(1)_Q$ with background charge; the other U(1) boson of the model parametrized by χ is compactified on a circle of radius $\sqrt{k}$.

(iii) W_k: This model describes semi-classically the well-known (semi)-wormhole background, which is an axionic instanton with manifest space-time supersymmetry. We have a conformally flat metric, a dilaton, as well as an anti-symmetric tensor field given by

$$\begin{aligned}
\frac{1}{k}ds^2 &= \frac{dud\bar{u}+dwd\bar{w}}{u\bar{u}+w\bar{w}}\,, \qquad -2\Phi = \log(u\bar{u}+w\bar{w})\,, \\
H &= \frac{1}{2(u\bar{u}+w\bar{w})^2}\left((\bar{u}du-ud\bar{u})\wedge dw\wedge d\bar{w}+(\bar{w}dw-wd\bar{w})\wedge du\wedge d\bar{u}\right). \quad (4.10)
\end{aligned}$$

Introducing real coordinates

$$u = e^{\rho+i\phi}\cos\theta\,, \qquad w = e^{\rho+i\psi}\sin\theta\,, \tag{4.11}$$

we obtain the equivalent expressions

$$\begin{aligned}
\frac{1}{k}ds^2 &= d\rho^2+d\theta^2+\sin^2\theta d\psi^2+\cos^2\theta d\phi^2\,, \\
-2\Phi &= 2\rho\,, \\
B_{\phi\psi} &= \cos^2\theta\,, \qquad \text{with} \quad H_{\theta\psi\phi} = \sin 2\theta\,. \quad (4.12)
\end{aligned}$$

Unlike the previous two backgrounds, W_k has a non-vanishing torsion that originates from the Wess-Zumino-Witten term of the $\mathrm{SU}(2)_k$ model. In this case, for large k, the three coordinates of $\mathrm{SU}(2)_k$ define a three-dimensional space with topology S^3, while the fourth coordinate ρ, which is non-compact with background charge, parameterizes the scale factor of the S^3 sphere, thus forming a four-dimensional (semi)-wormhole geometry as ρ varies along its throat; the throat becomes infinitely thin as $\rho\to-\infty$, where the dilaton field blows up. Also note that the dilaton is linear depending only on ρ and satisfies the axionic instanton condition

$$\frac{1}{2}H_{ijk} - \epsilon_{ijk}{}^{l}\partial_l\Phi = 0\,, \tag{4.13}$$

where $\epsilon_{ijk}{}^{l}$ is the covariantized fully anti-symmetric tensor in four dimensions; it supplies the vanishing condition for the supersymmetric variation of the dilatino field λ. Actually, W_k, which describes only the throat of the wormhole, can be promoted to the complete wormhole solution by performing a simple S-duality transformation that shifts $e^{2\Phi}$ by a constant without affecting the axionic instanton solution.

Finally, we note for completeness that C_k can be obtained from the (semi)-wormhole background by a T-duality transformation with respect to the Killing vector field ∂_ϕ of W_k, supplemented by a simple change of coordinates,

$$\phi = \tilde{\phi}-\frac{\tilde{\chi}}{2}\,, \qquad \chi = \tilde{\phi}+\frac{\tilde{\chi}}{2}\,. \tag{4.14}$$

(iv) *Fat throat model*: This model provides a generalization of the standard (semi)-wormhole solution with a throat that never becomes infinitely thin. As such, it may be viewed as a regularized version of W_k by switching on another moduli, say

r_0, which parameterizes the fatness of its throat. The fat throat model also satisfies the axionic instanton solution, thus providing explicit another example of an exact $N = (4,4)$ superconformal field theory block with $c = 4$ to all orders in α'. Furthermore, it can be related to the Δ_k model by a T-duality transformation, thus completing the picture of interrelations among all supersymmetric backgrounds with exact conformal field theory description as Wess-Zumino-Witten models. In the thin throat limit, as $r_0 \to 0$, one recovers the W_k model, in which case the T-dual background becomes C_k, as has been advocated above. Put it differently, the fat thoat model is the most general axionic instanton background that all other exact $N = (4,4)$ Wess-Zumino-Witten models relate to it by dualities or contractions in its moduli space.

More precisely, let us start from the bell-trumpet version of the model Δ_k and introduce the change of coordinates as in equation (4.14) above. The bell-black-hole version can also be treated along similar lines as described below. Performing a T-duality transformation with respect to the Killing vector field $\partial_{\tilde{\phi}}$, one obtains the following background that also includes a non-trivial anti-symmetric tensor field:

$$\begin{aligned} ds^2 &= d\rho^2 + d\theta^2 + \frac{1}{\cosh^2\rho - \cos^2\theta}\left(\cos^2\theta\cosh^2\rho d\tilde{\phi}^2 + \sin^2\theta\sinh^2\rho d\tilde{\chi}^2\right), \\ e^{-2\Phi} &= e^{-2\Phi_0}\left(\cosh^2\rho - \cos^2\theta\right), \\ B_{\tilde{\chi}\tilde{\phi}} &= \frac{\sin^2\theta\cosh^2\rho}{\cosh^2\rho - \cos^2\theta}, \end{aligned} \tag{4.15}$$

where the dilaton is determined up to a constant term written as $\exp(-2\Phi_0)$. The background we have obtained in this way is the fat throat model and it can be easily checked that it satisfies the axionic instanton condition [36, 37, 38].

In order to describe the details of the geometry, as well as exhibit its complex structure, it is useful to introduce the complex coordinates

$$u = r_0 \sin\theta \sinh\rho e^{i\tilde{\chi}}, \qquad w = r_0 \cos\theta\cosh\rho e^{i\tilde{\phi}}, \tag{4.16}$$

where $r_0^2 = \exp(-2\Phi_0)$, but it can also be normalized to 1. Then, following [38], the background assumes the conformally flat form

$$ds^2 = e^{2\Phi}(dud\bar{u} + dwd\bar{w}), \qquad e^{-2\Phi} = \sqrt{\left(|u|^2 + |w|^2 + r_0^2\right)^2 - 4r_0^2|w|^2}, \tag{4.17}$$

whereas the anti-symmetric tensor can also be written in complex notation using the axionic instanton condition. As for all axionic instantons, we have that $\exp(2\Phi)$ satisfies the Laplace equation in flat u, w coordinates. The model we are describing here reduces to the usual (semi)-wormhole space W_k by taking the limit $r_0 \to 0$, provided that a rescaling of the target space coordinates by r_0 is also taken into account; this rescaling means that the geometry looks like an ordinary (semi)-wormhole from far away. W_k has an O(4) symmetry which is broken to SO(2) $\times$ SO(2) when the moduli r_0 is turned on, or else by looking close enough to the details of the new background. It can be readily seen for $r_0 \neq 0$ that the dilaton blows up when $u = 0$ and $|w|^2 = r_0^2$, i.e., the singularities of the metric are not concentrated on a single point but they rather spread out on a ring of radius r_0. As a result, the background geometry, which still represents a (semi)-wormhole, has a fat throat that can only become infinitely thin when $r_0 \to 0$. Also, reversing the T-duality transformation that gave rise to the fat throat model, we find that the model C_k arises instead of Δ_k in the contraction limit $r_0 \to 0$. Finally, we add for completeness that in the other extreme limit, $r_0 \to \infty$, the axionic instanton becomes simplified by suitable rescaling of the coordinates; thus by zooming at its ring structure we arrive at a simpler supersymmetric background, which is actually dual to the trivial flat space solution F_4.

Summarizing the present exposition, we note that *all exact $N = (4, 4)$ superconformal field theories at hand with $c = 4$, are either axionic instantons or T-dual faces of them.* Of course, more general axionic instantons could also be used in the transverse space by appealing even to other solutions with conformally hyper-Kahler spaces and torsion , and their possible T-dual versions, but in those cases there is no exact conformal field theory description as Wess-Zumino-Witten models. An exact description is actually necessary in order to demonstrate explicitly the superconformal properties of such backgrounds, and hence obtain a realization of the extended world-sheet supersymmetry that is important for the light-cone formulation of string theory. Is is useful to recall at this point that all axionic instanton backgrounds exhibit manifest space-time supersymmetry as the dilatino variation vanishes and Killing spinors can be constructed explicitly by solving the vanishing condition of the gravitino variation. In these cases, the world-sheet supersymmetry is locally realized and the usual theorems that relate it with space-time supersymmetry are valid. On the other hand, T-duality with respect to non-triholomorphic isometries, like ∂_ϕ that was used in the examples above, does not commute with space-time supersymmetry, in the way that this is usually realized in the lowest order effective theory [39]. As a result, the dual geometries, which only have a dilaton but no torsion fields, cannot be space-time supersymmetric because the axionic instanton condition is violated after the T-duality transformation, [20, 36].

This is not surprising as T-dual backgrounds generically exhibit less symmetries than the original more symmetric geometries; in fact, only those isometries that commute with a given Killing vector field are manifest in the T-dual face of any model, whereas the remaining appear to be lost in the lowest order effective geometry. However, since T-duality always yields an equivalent string theory with the same amount of world-sheet supersymmetry, a paradox seems to appear with regard to the supersymmetric properties of the corresponding models. Its resolution is provided by the fact that the world-sheet superconformal algebra simply changes realization from local to non-local, thus evading the usual theorems that relate world-sheet with target space symmetries [20]. In the exact models we have presented above, it is possible to use the (non-local) parafermion fields of the Δ_k and C_k spaces to provide the required representation of the $N = (4, 4)$ world-sheet supersymmetry inspite of the lack of manifest space-time supersymmetry [19]. Since we will be mainly interested in world-sheet actions in the light-cone gauge of string theory for all backgrounds of pp-wave type, these remarks are certainly important for understanding the supersymmetric properties of the resulting interacting string models. Thus, in general, it

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will make sense to search for supersymmetric pp-wave backgrounds by solving the Killing spinor equations only in those case that the transverse space is replaced by an axionic instanton background, but not by other spaces.

5. R-R forms from $N = (4, 4)$ superconformal blocks

In this section we consider the case where four of the transverse coordinates are replaced by an exact $N = (4, 4)$ superconformal field theory block, which can be either Δ_k, C_k or W_k or its fat throat generalization, together with the dilaton and anti-symmetric NS-NS tensor fields asscociated with them. The remaining four transverse dimensions will be provided trivially by the flat euclidean space F_4, which will be parametrized using the complex coordinates v and z. Of course, one may consider more complicated possibilities where the remaining four flat coordinates are also replaced by any exact superconformal block with $N = (4, 4)$ world-sheet supersymmetry, in all possible combinations, but such hybrid models will not be worked out in detail here, with the exception of a short discussion at the end of this paper.

In any case, the results we describe in this section will be derived by solving the second order equations (3.7) for appropriate (but educated) choices of the front factor F, which give rise to some interesting integrable systems in the two-dimensional light-cone action of string theory. Hence, in this section, we will not rely on supersymmetric conditions, i.e., the existence of Killing spinors in target space, which typically provide additional relations between the front factor F and the R-R 5-form $\mathcal{F} = dx^+ \wedge \varphi$, and which are associated to first order equations. The main reason is that the models Δ_k and C_k do not exhibit space-time supersymmetry, since their $N = (4, 4)$ world-sheet supersymmetry is non-locally realized. On the other hand, the (semi)-wormhole models have space-time supersymmetry, as all axionic instanton backgrounds do. Here, we will only consider them briefly without being concerned about space-time supersymmetry of the resulting pp-wave configurations.

In this section, we will derive some explicit pp-wave solutions and discuss in detail their interpretation as integrable systems in the light-cone formulation of string theory. In one instance we will demonstrate explicitly the emergence of the complex sine-Gordon model, which can be viewed as integrable perturbation of the corresponding superconformal building blocks of the transverse space. In other cases we will arrive at the supersymmetric Liouville theory or at the complex sine-Liouville model. Thus, some old results about integrable perturbations of two-dimensional conformal field theories can be revisited and reinterpreted in the target space of string theory due to the presence of non-trivial R-R fields. We will also examine some relations among the resulting pp-wave backgrounds of type-IIB theory by relying on the web of known dualities that interconnect all exact superconformal field theory blocks with $N = (4, 4)$ superconformal invariance and $c = 4$. The construction of supersymmetric solutions for the (semi)-wormhole models will be examined separately in section 6, where we will focus exclusively on the Killing spinor equations for the fields g, Φ, B and $\mathcal{F}$ in general axionic instanton backgrounds. Only then, we will be able to obtain a systematic generalization of the Maldacena-Maoz space-time supersymmetric framework in the presence of non-trivial dilaton and NS-NS anti-symmetric tensor fields.

5.1 Solutions for the Δ_k model

We start by considering solutions of the main second order equation (3.7) that relates F with φ for the coset model Δ_k. Straightforward calculation shows that in this background we obtain the following result

$$\begin{aligned}\nabla_i\left(e^{-2\Phi}\nabla^i F\right) &= h^{u\bar{u}}h^{w\bar{w}}\left(h^{u\bar{u}}\partial_u\partial_{\bar{u}}F + h^{w\bar{w}}\partial_w\partial_{\bar{w}}F + \partial_v\partial_{\bar{v}}F + \partial_z\partial_{\bar{z}}F\right) + \\ &+ \frac{1}{2}h^{u\bar{u}}h^{w\bar{w}}\left(w\partial_w F + \bar{w}\partial_{\bar{w}}F - u\partial_u F - \bar{u}\partial_{\bar{u}}F\right) \end{aligned} \tag{5.1}$$

for any generic choice of the front factor F. However, not any F will produce a quantity that can be written as the norm-squared of an anti-self-dual closed 4-form φ. We will describe an interesting class of solutions that arise when Δ_k replaces four of the transverse coordinates, using the following choice of front factors:

$$F = -A^2|u|^2 + B^2|w|^2 + C^2|v|^2 + D^2|z|^2\,. \tag{5.2}$$

Substituting into the general expression, we find, in particular, that the corresponding φ is determined by equation (3.7) to be

$$\frac{1}{2}|\varphi|^2 = h^{u\bar{u}}h^{w\bar{w}}(2A^2|u|^2 + 2B^2|w|^2 + C^2 + D^2 - A^2 \pm B^2)\,, \tag{5.3}$$

where $h^{u\bar{u}} = 1 - |u|^2$ and $h^{w\bar{w}} = |w|^2 \pm 1$ are the components of the inverse metric of the $\mathrm{SU}(2)/\,\mathrm{U}(1)$ and $\mathrm{SL}(2)/\,\mathrm{U}(1)$ cosets respectively; here, the $\pm$ signs refer to the axial (respectively vector) gauging of the $\mathrm{SL}(2)/\,\mathrm{U}(1)$ part that describes the geometry of a two-dimensional black-hole (respectively trumpet).

Choosing the real constants A, B, C and D so that $\Delta^2 \equiv C^2 + D^2 - A^2 \pm B^2 \geq 0$, we obtain a positive definite result for $|\varphi|^2$, and therefore a solution to equation (3.7) can be easily obtained by choosing the 4-form φ as the following sum of $(1,3)$-forms, plus their complex conjugates:

$$\begin{aligned}\varphi &= 2Au du\wedge d\bar{w}\wedge d\bar{v}\wedge d\bar{z} + 2Bw dw\wedge d\bar{u}\wedge d\bar{v}\wedge d\bar{z} + \\ &+\Delta dv\wedge d\bar{u}\wedge d\bar{w}\wedge d\bar{z} + \Delta dz\wedge d\bar{u}\wedge d\bar{w}\wedge d\bar{v} + \mathrm{cc}. \end{aligned} \tag{5.4}$$

This is an anti-self-dual closed 4-form that provides a solution to the equations of type-IIB supergravity in ten dimensions in the presence of a self-dual (but non-constant) 5-form $\mathcal{F}$, as required. Note that the component $\varphi_{w\bar{u}\bar{v}\bar{z}} \sim w$ vanishes for $w = 0$, which is possible only for the two-dimensional black-hole geometry; in this case the geometry is that of an infinitely long cigar with scalar curvature $R = 1/(|w|^2 + 1)$ that becomes maximal at its tip, where $w = 0$. For the trumpet geometry we have the constraint $|w|^2 \geq 1$, and hence w can never vanish. On the other hand, the component $\varphi_{u\bar{w}\bar{v}\bar{z}} \sim u$ always vanishes at the symmetric top point $u = 0$ of the bell that describes the geometry of the $\mathrm{SU}(2)/\,\mathrm{U}(1)$ coset, in which case the scalar curvature of the corresponding two-dimensional space, $R = -1/(1-|u|^2)$, also assumes its maximum value. Hence, for the bell-cigar geometry, the R-R 5-form $\mathcal{F}$ vanishes at the point $(u,w) = (0,0)$ provided that the constants are chosen so that $\Delta = 0$.

In the present case, the bosonic world-sheet action that describes the ten-dimensional string model in the light-cone gauge, $x^+ = \tau$, assumes the form

$$S_{\rm lc} = S_{\rm SU(2)/\,U(1)} + S^{(\pm)}_{\rm SL(2)/\,U(1)} + S_v + S_z \,, \tag{5.5}$$

where the individual terms on the world-sheet Σ are

$$\begin{aligned} S_{\rm SU(2)/\,U(1)} &= \int_\Sigma \left(\frac{\partial u \bar{\partial} \bar{u} + \partial \bar{u} \bar{\partial} u}{1-|u|^2} + A^2 |u|^2 \right) , \\ S^{(\pm)}_{\rm SL(2)/\,U(1)} &= \int_\Sigma \left(\frac{\partial w \bar{\partial} \bar{w} + \partial \bar{w} \bar{\partial} w}{|w|^2 \pm 1} - B^2 |w|^2 \right) , \\ S_v + S_z &= \int_\Sigma \left(\partial v \bar{\partial} \bar{v} + \partial \bar{v} \bar{\partial} v - C^2 |v|^2 \right) + \int_\Sigma \left(\partial z \bar{\partial} \bar{z} + \partial \bar{z} \bar{\partial} z - D^2 |z|^2 \right) \end{aligned} \tag{5.6}$$

and $\pm$ refer to the two possible inequivalent geometries of the non-compact coset. Clearly, S_v and S_z describe the action of the complex bosons v and z with positive mass-squared terms C^2 and D^2 respectively. The first two terms on the other hand, describe the semi-classical action of the SU(2)/ U(1) and SL(2)/ U(1) gauged Wess-Zumino-Witten models, which are also perturbed by potential terms of the form $|u|^2$ and $|w|^2$ respectively.

It is instructive to write down the contribution from the two coset models in real coordinates, choosing for definiteness the parameterization $u = \cos\theta \exp(i\phi)$ and $w = \cosh\rho \exp(i\chi)$ in the case of the bell-trumpet geometry. We have, in particular,

$$\begin{aligned} S_{\rm SU(2)/\,U(1)} &= \int_\Sigma \left(\partial\theta \bar{\partial}\theta + \cot^2\theta \partial\phi \bar{\partial}\phi + A^2 \cos^2\theta \right) , \\ S^{(-)}_{\rm SL(2)/\,U(1)} &= \int_\Sigma \left(\partial\rho \bar{\partial}\rho + \coth^2\rho \partial\chi \bar{\partial}\chi - B^2 \cosh^2\rho \right) , \end{aligned} \tag{5.7}$$

whereas the case of the bell-cigar geometry can be discussed in a similar fashion. It is quite interesting to note that both these two-dimensional theories are integrable and coincide with the so-called complex sine-Gordon and complex sinh-Gordon models respectively [40, 41]; see also [42] for a detailed description of their conservation laws, among other things, [43] for some quantum mechanical calculations of their spectrum and scattering properties, and [44] for the construction of some periodic solutions.

First of all, in order to get a feeling about the nature of the resulting two-dimensional models, we may set $\phi = 0 = \chi$, which is consistent with the two-dimensional field equations.[5] Then, we observe that $S_{\rm SU(2)/\,U(1)} + S^{(-)}_{\rm SL(2)/\,U(1)}$ simply becomes the action of the ordinary sine-Gordon model plus that of the sinh-Gordon model,

$$S' = \int_\Sigma \left(\partial\theta \bar{\partial}\theta + A^2 \cos^2\theta \right) + \int_\Sigma \left(\partial\rho \bar{\partial}\rho - B^2 \cosh^2\rho \right) , \tag{5.8}$$

[5]This choice, however, is not consistent with the string background equations in target space as they reduce the dimensionality of the supergravity background by two, thus turning the critical string theory at hand into a non-critical theory.

which, by the way, for the special choice of parameters $A = B$ coincides with the bosonic action of the $N = 2$ sine-Gordon model for the field $\theta + i\rho$ and holomorphic superpotential $\mathcal{W} = \cos(\theta + i\rho)$. Then, it becomes obvious that in the presence of the ϕ and χ components of the metric, the complete action $S_{\mathrm{SU}(2)/\,\mathrm{U}(1)}$ provides a generalization of the sine-Gordon model in terms of a complex field u with $|u|^2 \leq 1$; as a result, the configurations of the complex sine-Gordon model carry an additional U(1) charge, which is associated with the Noether current with respect to the symmetry $\phi \to \phi + a$ of the action. Likewise, the complete action $S^{(-)}_{\mathrm{SL}(2)/\,\mathrm{U}(1)}$ provides a generalization of the sinh-Gordon model in terms of a complex field w with $|w|^2 \geq 1$; its configurations also carry a U(1) charge due to the invariance of the action under $\chi \to \chi + a$.

The complete action $S_{\mathrm{lc}} = S_{\mathrm{SU}(2)/\,\mathrm{U}(1)} + S_{\mathrm{SL}(2)/\,\mathrm{U}(1)} + S_v + S_z$ can be made supersymmetric for $A = B$ by straightforward generalization of the supersymmetric treatment that has already been discussed in the literature for the complex sine-Gordon model [45]; in this case, the potential of the two-dimensional theory is

$$V \equiv F = -|u|^2 + |w|^2 + |v|^2 + |z|^2 \,, \tag{5.9}$$

where we have also set $A = B = C = D = 1$ (and hence $\Delta = 0$), which in turn can be derived from a holomorphic superpotential,

$$\mathcal{W} = u + w + \frac{1}{2}v^2 + \frac{1}{2}z^2 \,, \tag{5.10}$$

as follows:

$$V = h^{u\bar{u}} \left| \frac{\partial \mathcal{W}}{\partial u} \right|^2 + h^{w\bar{w}} \left| \frac{\partial \mathcal{W}}{\partial w} \right|^2 + \left| \frac{\partial \mathcal{W}}{\partial v} \right|^2 + \left| \frac{\partial \mathcal{W}}{\partial z} \right|^2 . \tag{5.11}$$

Furthermore, the kinetic terms can be derived from the Kahler potential

$$\mathcal{K} = K_1(u, \bar{u}) + K_2(w, \bar{w}) + |v|^2 + |z|^2 \,, \tag{5.12}$$

where the Kahler potential terms of each coset model are defined separately via the equations

$$\partial_t K_1(t) = \frac{1}{t} \log \frac{1}{1-t} \qquad \text{for} \quad t = |u|^2 \,, \qquad \partial_s K_2(s) = \frac{1}{s} \log \frac{1}{s-1} \qquad \text{for} \quad s = |w|^2 \,. \tag{5.13}$$

The complex sine-Gordon model was originally introduced by Lund and Regge in their classical treatment of vortices [40], but it also has two other interesting interpretations that clarify its integrability properties. The first one views this model as a reduced O(4) non-linear sigma-model by exploiting the classical conformal invariance of the latter [41]; as such, it generalizes the derivation of the ordinary sine-Gordon model from the O(3) (i.e., CP^1) non-linear sigma-model by a similar type of reduction. In this case, the coupling constant of the potential term turns out to be positive, exactly as it appears in the action term $S_{\mathrm{SU}(2)/\,\mathrm{U}(1)}$ that has been obtained above with parameter $A^2 > 0$.[6] Likewise, the

[6]Note at this end that there are two physically inequivalent regions of the coupling constant in the complex sine-Gordon model depending on the sign of A^2; we cannot simply change the sign of the coupling constant in the field equations by redefinition of θ and ϕ, as in the case of the ordinary sine-Gordon model where $\theta \to \pi/2 - \theta$ does this job.

sinh-Gordon model, which appears in the action term $S_{\mathrm{SL}(2)/\,\mathrm{U}(1)}$, can arise as a reduced $\mathrm{O}(2,2)$ non-linear sigma model. Therefore, due to this description, it is natural to expect that massive mirror symmetry, which relates the $N=2$ sine-Gordon model to the deformed CP^1 (or sausage) model [46, 47], may have a suitable generalization in the present type-IIB string background using a deformed $\mathrm{O}(4)$ sausage model. We expect to return to this point elsewhere by studying the more general relation between multi-component sine-Gordon models on one hand, which arise classically as reduced non-linear sigma-models (or as perturbed Wess-Zumino-Witten models [48]), and their deformed (or sausage model) cousins on the other hand, which arise in quantum mechanical considerations by massive mirror symmetry.

A second interpretation of the complex sine-Gordon model (and its sinh-Gordon companion) can be found within the context of perturbed coset models of two-dimensional conformal field theory [42]. Recall that the primary fields of the $\mathrm{SU}(2)_k$ Wess-Zumino-Witten model are of the form $\Phi^{(j)}_{m,\bar{m}}$, where m and $\bar{m}$ run from $-j$ to j, each one taking $(2j+1)$ values, whereas j takes the values $0, 1/2, 1, \ldots, k/2$, [49, 50]. Their conformal dimensions are $j(j+1)/(k+2)$ and the quantum numbers m and $\bar{m}$ describe the $\mathrm{U}(1)$ charges of the primary fields in the two chiral sectors of the model. The simplest non-trivial such field is obtained for $j=1$, in which case its nine components can be naturally represented by the matrix elements

$$\Phi^{(1)}_{ab} = \mathrm{Tr}(g^{-1} T_a g T_b)\,, \tag{5.14}$$

where g, T are $\mathrm{SU}(2)$ group elements and Lie algebra generators respectively. Clearly, their conformal dimension becomes zero in the semi-classical limit $k \to \infty$. The $\mathrm{U}(1)$-neutral fields $\Phi^{(1)}_{m=0,\bar{m}=0}$ are also primary fields of the $\mathrm{SU}(2)_k/\,\mathrm{U}(1)$ coset model with the simplest example being the following $j=1$ neutral field from the complete list of nine fields $\Phi^{(1)}_{ab}$,

$$\epsilon_1 = \mathrm{Tr}(g^{-1}\sigma_3 g \sigma_3)\,. \tag{5.15}$$

This field can be identified with the so-called first thermal operator of the $\mathrm{SU}(2)_k/\,\mathrm{U}(1)$ coset model [50], which is usually denoted by ϵ_1, and its form can be worked out in terms of the target space coordinates in the semi-classical limit. It turns out to be [42]

$$\epsilon_1(u, \bar{u}) = 2(2|u|^2 - 1)\,, \tag{5.16}$$

and hence, modulo a shift in the vacuum energy, the perturbed conformal field theory of the $\mathrm{SU}(2)/\,\mathrm{U}(1)$ coset model,

$$S_{\mathrm{pert}} = S_{\mathrm{cft}} + \frac{A^2}{4}\int_{\Sigma} \epsilon_1\,, \tag{5.17}$$

can be readily seen to coincide with the the action of the complex sine-Gordon model, as it appears in our present construction of the light-cone string theory.

This is a perturbation that drives the conformal field theory away from criticality, but in a controlled way, as the resulting field theory is integrable both classically and quantum mechanically. It is quite interesting that such a perturbation (together with its sinh-Gordon companion) arise naturally in the context of type-IIB supergravity with non-trivial R-R 5-form. The particular choice of the front factor, which has been boldly made in the present construction, indeed describes an integrable perturbation of the superconformal building block Δ_k in the transverse space, as advertized. Note, however, that this is only the simplest example one may consider in the present framework, as there are other perturbations that can also appear in the light-cone formulation of the world-sheet action, which are described by the higher thermal operators ϵ_j of the coset model for appropriate choices of F, and hence $\mathcal{F}$. The details can be worked out by finding a description of the higher operators ϵ_j, which exist for all integer values j, in terms of the target space coordinates that conform the two-dimensional field theory solution into a type-IIB supergravity background.

Such generalizations will not be considered here, but we will only make a remark about their duality properties. There is the Krammers-Wannier duality that acts on the world-sheet by exchanging the spin variables σ with the dual spin variables μ of the coset model (order–disorder duality) [50]. Under this action, the thermal operators behave as $\epsilon_j \to (-1)^j \epsilon_j$, and hence for odd values of j they flip sign. It is clear from the structure of our solution, which corresponds to $j = 1$, that the Krammers-Wannier duality cannot be interpreted as a duality symmetry within type-IIB supergravity because it amounts to changing A^2 to $-A^2$ in the expression for the front factor F; such a change is quite severe, as it can be easily seen that prevents the existence of solutions for the 4-form φ in our class of models. Hence, the positivity of the coupling constant A^2 is a very rigid requirement here that does not permit the use of the Krammers-Wannier duality in the context of the target space theory. One possibility to have a true manifestation of this duality within the context of type-IIB supergravity is to consider solutions with front factors F that correspond to perturbations by ϵ_j with even values of j. Thus, it will be interesting to explore further the existence of such solutions and obtain the corresponding R-R 5-forms, if they can be defined consistently.

5.2 Solutions for the C_k model

Next, we consider some interesting solutions that arise when four of the transverse coordinates are replaced by the geometry of the C_k conformal field theory in the presence of the appropriate dilaton field. Straightforward calculation shows that for generic choices of the front factor F, we have the result

$$\begin{aligned}\nabla_i \left(e^{-2\Phi} \nabla^i F\right) &= e^{w+\bar{w}} h^{u\bar{u}} \left(h^{u\bar{u}} \partial_u \partial_{\bar{u}} F + \partial_w \partial_{\bar{w}} F + \partial_v \partial_{\bar{v}} F + \partial_z \partial_{\bar{z}} F\right) + \\ &\quad + \frac{1}{2} e^{w+\bar{w}} h^{u\bar{u}} \left(\partial_w F + \partial_{\bar{w}} F - u\partial_u F - \bar{u}\partial_{\bar{u}} F\right). \end{aligned} \tag{5.18}$$

Thus, making the special choice of front factor

$$F = A^2 e^{w+\bar{w}} - B^2 e^{w+\bar{w}} |u|^2 - C^2 |u|^2 + D^2 |v|^2 + E^2 |z|^2 \,, \tag{5.19}$$

we find, in particular, that the 4-form φ has to satisfy the equation

$$\frac{1}{2} |\varphi|^2 = h^{u\bar{u}} e^{w+\bar{w}} \left((2A^2 - B^2) e^{w+\bar{w}} + 2C^2 |u|^2 + D^2 + E^2 - C^2\right). \tag{5.20}$$

Setting $C^2 = 2D^2 = 2E^2$ in the sequel, in order to simplify the presentation, we may solve equation (3.7) by expressing the 4-form φ as a suitable sum of $(1,3)$-forms, plus their complex conjugates, provided that $2A^2 - B^2 \geq 0$. Another choice of parameters that will be useful later, while comparing with solutions of the W_k space, is $D^2 = -E^2$, which also requires $C = 0$ for positive definiteness of $|\varphi|^2$; actually, this choice can also be incorporated in case (iii) below, provided that the appropriate change of parameters takes place. We also note for completeness that adding a term proportional to $|w|^2$ in F appears to be inconsistent with the existence of a closed anti-self-dual 4-form φ in the present background.

In the following, we investigate some special cases separately that provide light-cone systems with increasing degree of complexity.

(i) $A = B = 0$: The solution for the 4-form is

$$\varphi = 2Cue^{\bar{w}} du \wedge d\bar{w} \wedge d\bar{v} \wedge d\bar{z} + \text{cc}. \tag{5.21}$$

which vanishes at the symmetric point $u = 0$ of the bell geometry $\mathrm{SU}(2)/\mathrm{U}(1)$. Then, it is straightforward to see that the world-sheet action in the light-cone formulation of the ten dimensional model becomes

$$\begin{aligned} S_{\text{lc}} = & \int_\Sigma \left(\frac{\partial u \bar{\partial} u + \partial \bar{u} \bar{\partial} u}{1 - |u|^2} + C^2 |u|^2 \right) + \int_\Sigma \left(\partial w \bar{\partial} \bar{w} + \partial \bar{w} \bar{\partial} w \right) + \\ & + \int_\Sigma \left(\partial v \bar{\partial} \bar{v} + \partial \bar{v} \bar{\partial} v + \partial z \bar{\partial} \bar{z} + \partial \bar{z} \bar{\partial} z - \frac{1}{2} C^2 (|v|^2 + |z|^2) \right). \end{aligned} \tag{5.22}$$

It is thus clear that apart from the three complex bosons w, v and z with their respective mass terms, we also have the contribution of a complex sine-Gordon model associated to the field u that parametrizes the $\mathrm{SU}(2)/\mathrm{U}(1)$ part of the building block C_k. We may refer to the previous discussion for the properties of this integrable system.

(ii) $B = 0$: In this case the 4-form can be chosen as

$$\varphi = 2Ae^{2w} dw \wedge d\bar{u} \wedge d\bar{v} \wedge d\bar{z} + 2Cue^{\bar{w}} du \wedge d\bar{w} \wedge d\bar{v} \wedge d\bar{z} + \text{cc}. \tag{5.23}$$

and the light-cone action becomes,

$$\begin{aligned} S_{\text{lc}} = & \int_\Sigma \left(\frac{\partial u \bar{\partial} u + \partial \bar{u} \bar{\partial} u}{1 - |u|^2} + C^2 |u|^2 \right) + \int_\Sigma \left(\partial w \bar{\partial} \bar{w} + \partial \bar{w} \bar{\partial} w - A^2 \mid e^w \mid^2 \right) + \\ & + \int_\Sigma \left(\partial v \bar{\partial} \bar{v} + \partial \bar{v} \bar{\partial} v + \partial z \bar{\partial} \bar{z} + \partial \bar{z} \bar{\partial} z - \frac{1}{2} C^2 (|v|^2 + |z|^2) \right). \end{aligned} \tag{5.24}$$

The first term corresponds again to the complex sine-Gordon model, whereas the second term describes the bosonic part of the $N = 2$ Liouville theory with holomorphic superpotential $\mathcal{W} = \exp w$ [51]; as for the last two terms, they describe again free complex bosons with their respective mass terms. Clearly, for $A = 0$, the Liouville potential vanishes and one recovers case (i) above. Also, C can take any value, including zero, in which case the total potential simplifies to the Liouville term alone.

In any case, for $B = 0$, the resulting solution involves the complex sine-Gordon and the Liouville models, which are decoupled from each other. We will present next another choice of parameters that leads to a coupled complex sine-Liouville model on the world-sheet.

(iii) $C = 0$: In this case we must also impose the restriction $2A^2 \geq B^2$ in order to obtain a real solution for the 4-form φ. Then, we have,

$$\varphi = \sqrt{2(2A^2 - B^2)}e^{2w} dw \wedge d\bar{u} \wedge d\bar{v} \wedge d\bar{z} + \text{cc}. \tag{5.25}$$

and the corresponding light-cone sigma-model action takes the form

$$S_{\text{lc}} = \int_\Sigma \left(\frac{\partial u \bar{\partial} u + \partial \bar{u} \bar{\partial} u}{1 - |u|^2} + \partial w \bar{\partial} \bar{w} + \partial \bar{w} \bar{\partial} w - \mid e^w \mid^2 (A^2 - B^2 |u|^2) \right) + \\ + \int_\Sigma \left(\partial v \bar{\partial} \bar{v} + \partial \bar{v} \bar{\partial} v + \partial z \bar{\partial} \bar{z} + \partial \bar{z} \bar{\partial} z \right). \tag{5.26}$$

For $B = 0$, we recover a special case of the solution (ii) above, where there is only a Liouville potential for the field w. For $2A^2 \geq B^2 \neq 0$, however, we obtain a coupled system of the complex sine-Gordon and Liouville models, which can be thought as a complex field generalization of the usual sine-Liouville model. To see this, it is instructive to use the parametrization of the C_k model in terms of real variables and set $\phi = \chi = 0$[7] in the notation that we introduced in the previous section. Then, the (u, w)-dependent part of the two-dimensional action assumes the form

$$S' = \int_\Sigma \left(\partial \theta \bar{\partial} \theta + \partial \rho \bar{\partial} \rho - A^2 e^{2\rho} \sin^2 \theta \right), \tag{5.27}$$

which is a suitable form of the sine-Liouville model that has been written here for the special choice of parameters $A = B$ (see, for instance, [52] and references therein); note that, in general, for more arbitrary values of the parameters, we will also have the appearance of a two dimensional cosmological constant term in the action.

It seems that the complex sine-Liouville model that we encounter here cannot be made supersymmetric as it stands for any choice of A and B, as the potential term cannot be obtained from a holomorphic superpotential in u and w. We note for completeness that a supersymmetric version of such models, which has been known for some time, is provided by the complex sinh-Gordon-Liouville theory,

$$\tilde{S}' = \int_\Sigma \left(\frac{\partial u \bar{\partial} u + \partial \bar{u} \bar{\partial} u}{|u|^2 - 1} + \partial w \bar{\partial} \bar{w} + \partial \bar{w} \bar{\partial} w - A^2 \mid e^w \mid^2 (2|u|^2 - 1) \right), \tag{5.28}$$

where $|u| \geq 1$, i.e., for a non-compact version of C_k, where the $\mathrm{SU}(2)/\mathrm{U}(1)$ coset is replaced by the euclidean trumpet coset $\mathrm{SL}(2)/\mathrm{U}(1)$. The latter version is known to coincide with the called non-abelian B_2 Toda system, [53], and its potential can be easily derived from the holomorphic superpotential $\mathcal{W} = u \exp w$.

[7] As before, this choice is only made here for illustrative purposes, as it is not consistent with the string background equations in the critical dimension $D = 10$.

5.3 Solutions for the W_k model

Next in our list of examples, we consider solutions where the superconformal field theory block W_k replaces four of the transverse space coordinates in ten-dimensional type-IIB supergravity. The (semi)-wormhole background, being an axionic instanton, possesses space-time supersymmetry and therefore it can admit a large class of supersymmetric solutions with non-trivial R-R 5-form and front factor F. For the moment being, we will construct a particular set of simple solutions for the (semi)-wormhole geometry by examining the second order string background equations (3.7), and also try to connect them with the solutions of the C_k model via T-duality. Later, we will revisit this model from a more general viewpoint that provides solutions to its Killing spinor equations.

It is straightforward to verify in the present case that

$$\nabla_i \left(e^{-2\Phi}\nabla^i F\right) = h^{u\bar{u}}h^{w\bar{w}}\left(\partial_u\partial_{\bar{u}}F + \partial_w\partial_{\bar{w}}F\right) + e^{-2\Phi}\left(\partial_v\partial_{\bar{v}}F + \partial_z\partial_{\bar{z}}F\right) \tag{5.29}$$

for all possible choices of the front factor F. Here, we only consider front factors which are quadratic in the target space coordinates, namely,

$$F = A^2|u|^2 + B^2|w|^2 + C^2|v|^2 - D^2|z|^2\,. \tag{5.30}$$

Explicit calculation shows in this case that

$$\frac{1}{2}|\varphi|^2 = (A^2+B^2)\left(|u|^2+|w|^2\right)^2 + (C^2-D^2)\left(|u|^2+|w|^2\right) \tag{5.31}$$

and therefore, since $h^{u\bar{u}} = h^{w\bar{w}} = |u|^2+|w|^2 = \exp(-2\Phi)$, we conclude that a solution to second order equation (3.7) can be easily obtained provided that $C = D$. With this choice of parameters, we find that φ can be written as the following sum of constant $(1,3)$-forms, plus their complex conjugates:

$$\varphi = \sqrt{2}A du\wedge d\bar{w}\wedge d\bar{v}\wedge d\bar{z} + \sqrt{2}B dw\wedge d\bar{u}\wedge d\bar{v}\wedge d\bar{z} + \text{cc.} \tag{5.32}$$

Then, the world-sheet action in the light-cone gauge of the corresponding string background assumes a form, which is conveniently written here using the real coordinates of W_k,

$$\begin{aligned} S_{\rm lc} = &\int_\Sigma \left(\partial\rho\bar{\partial}\rho + \partial\theta\bar{\partial}\theta + \sin^2\theta\partial\psi\bar{\partial}\psi + \cos^2\theta\left(\partial\phi\bar{\partial}\phi + \partial\phi\bar{\partial}\psi - \bar{\partial}\phi\partial\psi\right)\right) - \\ &- \int_\Sigma e^{2\rho}(A^2\cos^2\theta + B^2\sin^2\theta) + \\ &+ \int_\Sigma \left(\partial v\bar{\partial}\bar{v} + \partial\bar{v}\bar{\partial}v + \partial z\bar{\partial}\bar{z} + \partial\bar{z}\bar{\partial}z - C^2|v|^2 + C^2|z|^2\right). \end{aligned} \tag{5.33}$$

Thus, we see the emergence of a Liouville term for the field ρ that originates from the non-compact boson $\mathrm{U}(1)_Q$ of the model $W_k = \mathrm{SU}(2)_k\times\mathrm{U}(1)_Q$ with background charge, which is also coupled to θ for $A\neq B$, as in the sine-Liouville model. The first term in $S_{\rm lc}$ is the usual semi-classical action of the $\mathrm{SU}(2)\times\mathrm{U}(1)$ Wess-Zumino-Witten model, where we have also included the contribution from the anti-symmetric tensor field $B_{\phi\psi} = \cos^2\theta$, whereas the last term describes two complex bosons v, z with mass-squared terms, one of which is tachyonic; these mass terms can also be set equal to zero without drastic change of our solution.

Note that in heterotic and type-II string theory, the five-brane solution is constructed by considering the ten-dimensional background $M_6 \times W_k$, where M_6 is the six-dimensional Minkowski space. Here, we are making another practical use of the (semi)-wormhole geometry by embeding it into a ten-dimensional background of type-IIB supergravity of the form $\tilde{P}_6 \times W_k$, where

$$ds^2(\tilde{P}_6) = -4dx^+dx^- - \left(A^2|u|^2 + B^2|w|^2 + C^2(|v|^2 - |z|^2)\right)\left(dx^+\right)^2 + dvd\bar{v} + dzd\bar{z} \tag{5.34}$$

is the six-dimensional part of a pp-wave background that gives rise to a non-trivial constant RR 5-form $\mathcal{F}$. It is worth mentioning that for $A = 0 = B$, the space $\tilde{P}_6$ becomes the standard pp-wave backgroung in six dimensional space-time, P_6, having the usual quadratic front factor in v and z with zero trace; if we also have $C = 0$, which is certainly allowed without threatening our final result with trivialization, P_6 will be the Minkowski space M_6 written in light-cone coordinates. Then, for $A, B \neq 0$, we are perturbing the front factor of the the six-dimensional metric by the term $A|u|^2 + B^2|w|^2$, which depends on the remaining (semi)-wormhole coordinates, thus producing the relevant deformation of the pp-wave background to $\tilde{P}_6$.

One may inquire at this point about the effect of the T-duality transformation with respect to the isometry generated by the Killing vector field ∂_ϕ, which relates the superconformal field theory W_k to C_k. Certainly, this has no effect on the combination $A^2|u|^2 + B^2|w|^2 + C^2(|v|^2 - |z|^2) \equiv e^{2\rho}(A^2\cos^2\theta + B^2\sin^2\theta) + C^2(|v|^2 - |z|^2)$ that provides the front factor for the (semi)-wormhole background in transverse space; therefore, it can be used without change in order to describe solutions of type-IIB supergravity in ten dimensions, but with the superconformal field theory block C_k living now in their transverse space. Writing the ρ-dependence of F in terms of the new complex coordinates u and w that parameterize C_k, which should be distinguished from the corresponding complex coordinates that parameterize W_k, it is easy to note that this front factor takes the following form in the new (dual) background:

$$F = B^2e^{w+\bar{w}} - (B^2 - A^2)e^{w+\bar{w}}|u|^2 + C^2(|v|^2 - |z|^2)\,. \tag{5.35}$$

This corresponds precisely to the general choice of front factor that was earlier made for the model C_k, provided that one sets there $C = 0$ and rename $D^2 = -E^2$ as the new C^2; we also have to rename A^2 and B^2 by comparing the two different expressions, in which case the condition $2A^2 \geq B^2$ for the C_k models becomes $A^2 + B^2 \geq 0$ for W_k. Since the dilaton also changes under the proposed T-duality, one concludes that the operation of duality in transverse space yields the R-R 5-form of the C_k background as described by the special class of solutions (iii) in the corresponding complex variables and for the appropriate choice of parameters that turn the z-boson into a tachyon with mass-squared equal to $-C^2$.

Thus, although T-duality relates in general type-IIB to type-IIA backgrounds in ten dimensions, we see here that it can also be used as a trick in the transverse space to provide a solution generating technique for type-IIB supergravity with non-trivial R-R 5-form fields. By the same token, we may attempt to find other type-IIB backgrounds for the W_k model, which are obtained by duality from the more general class of C_k models that have been studied above. Writing the front factor of these C_k models in terms of real coordinates and then reexpressing them in terms of the complex coordinates of the W_k model we find the following result,

$$F = A^2 \left(|u|^2 + |w|^2\right) - B^2|u|^2 - C^2\frac{|u|^2}{|u|^2 + |w|^2} + D^2|v|^2 + E^2|z|^2 . \tag{5.36}$$

In order to examine the existence of solutions of type IIB supergravity for the W_k model in transverse space, we apply the general formula (5.29) for this particular choice of front factor, and find

$$\nabla_i \left(e^{-2\Phi}\nabla^i F\right) = (2A^2 - B^2) \left(|u|^2 + |w|^2\right)^2 + 2C^2|u|^2 . \tag{5.37}$$

Here, we have also set for convenience $C^2 = 2D^2 = 2E^2$, as in the corresponding C_k models with $2A^2 \geq B^2$.

We observe that a 4-form φ exists for $C = 0$, leading to a consistent type-IIB background for W_k with quadratic front factor, as before. For $C \neq 0$, however, there is no consistent solution of type-IIB theory with an R-R 4-form φ; this is not surprising as T-duality is not always meant to act consistently within type-IIB theory, but rather relate IIA to IIB. In conclusion, only in certain cases the different classes of pp-wave solutions of the C_k and W_k models can map to each other by T-duality.

5.4 Solutions for the fat throat model

In this case we find again for any front factor that

$$\nabla_i \left(e^{-2\Phi}\nabla^i F\right) = h^{u\bar{u}}h^{w\bar{w}} \left(\partial_u\partial_{\bar{u}}F + \partial_w\partial_{\bar{w}}F\right) + e^{-2\Phi} \left(\partial_v\partial_{\bar{v}}F + \partial_z\partial_{\bar{z}}F\right) \tag{5.38}$$

taking into account the corresponding expressions for the metric and dilaton fields, which also satisfy $h^{u\bar{u}} = h^{w\bar{w}} = \exp(-2\Phi)$. We can repeat the previous analysis and find solutions of the second order equation (3.7) for some simple, e.g., quadratic choices of the front factor. However, since in the next section we will describe the general class of supersymmetric pp-wave solutions for all axionic instanton backgrounds, including the fat throat model, we will confine our analysis here only to the construction of those solutions which can be obtained by T-duality from the special solutions that are already available for the Δ_k model. We will find once more that T-duality cannot always act as a solution generating symmetry within type-IIB theory alone.

Let us start by considering front factors of the Δ_k model written in real coordinates as $F = -A^2 \sin^2\theta + B^2 \sinh^2\rho + C^2|v|^2 + D^2|z|^2$. T-duality with respect to ∂_ϕ does not affect F as it stands, but in the fat throat model that results by duality the functional dependence of F on the new complex coordinates will be different. Using the appropriate change of coordinates, we find the following expression for the fat throat model:

$$F = \frac{1}{2r_0^2}(A^2+B^2)\left(|u|^2+|w|^2-r_0^2\right)+C^2|v|^2+D^2|z|^2 - \\ - \frac{1}{2r_0^2}(A^2-B^2)\sqrt{\left(|u|^2+|w|^2+r_0^2\right)^2-4r_0^2|w|^2}\,. \tag{5.39}$$

Therefore, to determine φ, we first compute in this case

$$\nabla_i\left(e^{-2\Phi}\nabla^i F\right) = h^{u\bar{u}}h^{w\bar{w}}\frac{1}{r_0^2}(A^2+B^2)+e^{-2\Phi}\left(C^2+D^2+\frac{1}{r_0^2}(B^2-A^2)\left(|u|^2+|w|^2\right)\right). \tag{5.40}$$

We see that only if $A^2 = B^2$ and $C^2+D^2 = 0$, we will have a consistent solution for the corresponding φ in the fat throat model in terms of $(1,3)$ forms and their complex conjugates. Otherwise, there will be no type IIB background with a fat throat placed in its transverse space, which is T-dual to Δ_k models for this particular class of front factors.

Conversely, as soon as we can find an entire class of supersymmetric solutions for the fat throat model, as for all other axionic instantons, we may try to build consistent type-IIB backgrounds for all other spaces that can be obtained from it by T-duality (and in some cases supplemented by a contraction with respect to r_0), provided that its action is consistently implemented within the type-IIB framework. This part will not be analyzed here, but certainly it has the potential to produce some interesting models, which will be very different in nature from the complex sine-Gordon model. We also leave this construction to future work.

6. Solutions with space-time supersymmetry

In this section we focus only on axionic instanton solutions as building blocks of the transverse space of gravitational backgrounds of pp-wave type. First, we show that the second order equation (3.7) for the front factor F and the anti-self-dual 4-form φ can be easily mapped to the equation that defines ordinary pp-wave solutions in flat transverse space by a conformal rotation of the target space metric; in this case, we consider backgrounds with front factor (and hence 4-form) that have support only on the four transverse space coordinates provided by any axionic instanton solution. This will also motivate the study of the Killing spinor equations that we undertake in the sequel to characterize all possible supersymmetric solutions by solving the Killing spinor equations. It will turn out that the supersymmetric solutions can be obtained systematically from the class of solutions that have already been described by Maldacena and Maoz in flat transverse space with no dilaton nor anti-symmetric tensor fields, provided that one takes into account the appropriate chiral projections imposed by the axionic instanton condition. The explicit construction of the supersymmetric solutions requires a number of steps, which will be outlined in the following subsections in all detail.

6.1 General structure of the solutions

Consider gravitational pp-wave backgrounds, where the front factor F as well as the 4-form φ have support on the transverse space coordinates, and further consider that φ_{mn}, $\varphi_{m\bar{n}}$ (and their complex conjugates) have a block-diagonal form, using the short-hand notation,

$$\varphi_{mn} = \begin{pmatrix} \star & \star & & \\ \star & \star & & \\ & & \star & \star \\ & & \star & \star \end{pmatrix}, \qquad \varphi_{m\bar{n}} = \begin{pmatrix} \star & \star & & \\ \star & \star & & \\ & & \star & \star \\ & & \star & \star \end{pmatrix}, \tag{6.1}$$

thus setting all components in the off-diagonal blocks equal to zero. Actually, at it has already been pointed out in subsection 3.2, this is in accordance with the 2+2 decomposition of the four space-time indices of φ, in the sense that the two first indices of the allowed $(1,3)$ forms $\varphi_{m\bar{n}\bar{p}\bar{q}}$, as well as of the $(2,2)$ forms $\varphi_{m\bar{n}p\bar{p}}$, take values in the axionic instanton block and the remaining two in the other. Notice that only in this case, *the anti-self-duality condition for the R-R 4-form, $\varphi = -^\star\varphi$, remains invariant with respect to conformal rescaling of each four-dimensional building block of the transverse space separately*, and hence the same components of φ can be used in any axionic instanton block as in flat space. Furthermore, the closure of such 4-forms provides the same conditions as in flat space because φ is totally anti-symmetric and covariant derivatives translate into ordinary derivatives. Thus, it is conceivable that flat space solutions can be lifted to the axionic instanton block by conformal rotation, which, however, may yield pp-waves with different front factor F, as the second order equation (3.7) that relates F to the norm of of the R-R form, i.e., $|\varphi|^2$, is metric dependent.

We can be more explicit by looking at the general structure of the main equation (3.7) in any four-dimensional axionic instanton block. Using the complex coordinates u and w (and their conjugates) to parameterize a general axionic instanton background with conformally flat metric, and v, z (and their conjugates) to parameterize the remaining flat directions, we find

$$\nabla_i \left(e^{-2\Phi}\nabla^i F\right) = e^{-4\Phi}(\partial_u\partial_{\bar{u}}F + \partial_w\partial_{\bar{w}}F) + e^{-2\Phi}(\partial_v\partial_{\bar{v}}F + \partial_z\partial_{\bar{z}}F)\,, \tag{6.2}$$

where

$$e^{-2\Phi} = h^{u\bar{u}} = h^{w\bar{w}}\,, \qquad \text{with} \quad (\partial_u\partial_{\bar{u}} + \partial_w\partial_{\bar{w}})e^{2\Phi} = 0\,. \tag{6.3}$$

Then, it is clear that pp-wave solutions can be easily constructed from all known pp-wave solutions in flat transverse space by making the appropriate choices that respect the anti-self-duality of φ, as well as its closure and co-closure in the new gravitational backgrounds. Note for this purpose that the norm of the 4-form, $|\varphi|^2$, which involves contractions of the transverse space indices, using the components of the inverse metric $h^{u\bar{u}}$ and $h^{w\bar{w}}$, will then naturally be related to the flat space norm by the conformal factor, since

$$|\varphi|^2 = e^{-4\Phi}\sum_{u,w}\left(|\varphi_{mn}|^2 + |\varphi_{m\bar{n}}|^2\right) + \sum_{v,z}\left(|\varphi_{mn}|^2 + |\varphi_{m\bar{n}}|^2\right), \tag{6.4}$$

which has to be compared with equation (6.2) above. Thus, solutions in an axionic instanton background can be easily mapped into flat space solutions by conformal transformations, provided that one considers front factors without v or z dependence, i.e., $F(u, w, \bar{u}, \bar{w})$, and likewise for φ, with $\varphi_{mn}(u, w, \bar{u}, \bar{w})$ and $\varphi_{m\bar{n}}(u, w, \bar{u}, \bar{w})$ satisfying the additional requirement that the indices m, n take values only in the first block, thus being zero otherwise. Clearly, the latter requirement eliminates the second bunch of terms in $|\varphi|^2$ that involve summation over v and z, as they do not depend on the conformal factor in the right way. Finally, we note for completeness that adding any real harmonic function of v, z (and their complex conjugates) to the front factor F can always yield more general pp-wave solutions; these generalizations will not be of interest to us, however, as they do not affect the R-R fields, which are inert to all such modifications.

This general observation, which applies to any solution of the second order equation (3.7), clearly also includes the complete list of supersymmetric solutions that have been constructed by Maldacena and Maoz in flat transverse space with the additional restriction that all data is appropriately confined in the axionic instanton block. Thus, one may expect that the solutions of the Killing spinor equations for an axionic instanton background in the presence of a R-R 5-form will also reduce to the corresponding supersymmetric solutions in flat transverse space, leading to new classes of space-time supersymmetric backgrounds. As we will see in the next subsections, which only concern the analysis of the supersymmetric configurations, this proceeds technically by making use of the axionic instanton condition to simplify the Killing spinor equations on pp-wave background and reduce them to a much simpler set of flat space conditions. However, due to the chiral properties of the Killing spinors imposed by the axionic instanton condition, only flat space solutions with $(1, 1)$ space-time supersymmetry will lead to supersymmetric pp-wave solutions for any axionic instanton block living in the transverse space. Thus, solutions with more space-time supersymmetry in flat space will only be solutions of the second order equations in the present case.

6.2 Supersymmetry variations

We start by writing the supersymmetry transformations of the type-IIB dilatino and gravitino fields, which are usually given in the Einstein frame by the equations, [24, 27],

$$\begin{aligned} \delta\lambda' &= i\Gamma'^{\mu}\mathcal{P}_{\mu}\epsilon'^{\star} - \frac{i}{24}\Gamma'^{\kappa\lambda\nu}\mathcal{G}_{\kappa\lambda\nu}\epsilon' \,, \\ \delta\Psi'_{\mu} &= D_{\mu}\epsilon' + \frac{1}{96}\left(\Gamma'^{\ \kappa\lambda\nu}_{\mu}\mathcal{G}_{\kappa\lambda\nu} - 9\Gamma'^{\lambda\nu}\mathcal{G}_{\mu\lambda\nu}\right)\epsilon'^{\star} + \\ &\quad + \frac{i}{16\cdot 5!}\Gamma'^{\kappa\lambda\nu\rho\sigma}\Gamma'_{\mu}\mathcal{F}_{\kappa\lambda\nu\rho\sigma}\epsilon' \end{aligned} \tag{6.5}$$

omiting the fermionic terms, which are all set equal to zero for bosonic solutions. Here, the prime refers to the Einstein frame variables, the star denotes complex conjugation, and the various quantities that appear in the variations above are defined to be

$$D_{\mu}\epsilon' = \left(\partial_{\mu} + \frac{1}{4}\omega'^{\ \nu\rho}_{\mu}\Gamma_{\nu\rho} - \frac{i}{2}\mathcal{Q}_{\mu}\right)\epsilon' \,,$$

$$
\begin{aligned}
\mathcal{Q}_\mu &= -i\epsilon_{\alpha\beta}V_-^\alpha \partial_\mu V_+^\beta \,, \\
\mathcal{P}_\mu &= -\epsilon_{\alpha\beta}V_+^\alpha \partial_\mu V_+^\beta \,, \\
\mathcal{G}_{\kappa\lambda\nu} &= -\epsilon_{\alpha\beta}V_+^\alpha F_{\kappa\lambda\nu}^\beta \,,
\end{aligned}
\tag{6.6}
$$

where α and β take the values 1 and 2, with $\epsilon_{12} = 1$, and $F^1_{\kappa\lambda\nu} = (F^2_{\kappa\lambda\nu})^\star$. These equations are written down using the SU(1,1) formulation of the theory, which relates $V_\pm^\alpha$ to the dilaton Φ and the R-R scalar $C^{(0)}$ by the equation

$$
\begin{pmatrix} 1 & 1 \\ & \\ -i & i \end{pmatrix}
\begin{pmatrix} V_-^1 & V_+^1 \\ & \\ V_-^2 & V_+^2 \end{pmatrix}
= e^{\Phi/2}
\begin{pmatrix} -\bar{\tau} & -\tau \\ & \\ 1 & 1 \end{pmatrix}
\tag{6.7}
$$

where $\tau = C^{(0)} + i\exp\Phi$. Furthermore, in the most general situation, the field strengths of the NS-NS and R-R 2-forms B and $C^{(2)}$ respectively are related to $F^\alpha_{\kappa\lambda\nu}$ as follows,

$$
\begin{pmatrix} -dC^{(2)} \\ \\ dB \end{pmatrix}
= \begin{pmatrix} \mathrm{Re}\, F^1 \\ \mathrm{Im}\, F^1 \end{pmatrix}
= \frac{1}{2}
\begin{pmatrix} 1 & 1 \\ & \\ -i & i \end{pmatrix}
\begin{pmatrix} F^1 \\ \\ F^2 \end{pmatrix} .
\tag{6.8}
$$

In this paper we focus on configurations with only a R-R 5-form turned on, setting the potentials of the remaining R-R fields equal to zero, i.e., $C^{(0)} = 0 = C^{(2)}$. In this case, we have the identifications

$$
\begin{aligned}
V_+^1 &= i\sinh\frac{1}{2}\Phi \,, \qquad & V_+^2 &= -i\cosh\frac{1}{2}\Phi \,, \\
V_-^1 &= i\cosh\frac{1}{2}\Phi \,, \qquad & V_-^2 &= -i\sinh\frac{1}{2}\Phi \,.
\end{aligned}
\tag{6.9}
$$

As a result, we obtain immediately that

$$
\begin{aligned}
\mathcal{Q}_\mu &= 0 \,, \qquad\qquad & \mathcal{P}_\mu &= \frac{1}{2}\partial_\mu\Phi \,, \\
\mathcal{G}_{\kappa\lambda\nu} &= e^{-\Phi/2}H_{\kappa\lambda\nu} & &
\end{aligned}
\tag{6.10}
$$

and the corresponding supersymmetry variations simplify considerably.

We will present the first order equations that arise from the vanishing condition of the dilatino and gravitino equations in the *sigma model frame*, which is more appropriate for describing our solutions. Since the two frames are related to each other by $g_{\mu\nu} = \exp(\Phi/2)g'_{\mu\nu}$, and hence $g^{\mu\nu} = \exp(-\Phi/2)g'^{\mu\nu}$, we have the relation among the gamma matrices $\Gamma^\mu = \exp(-\Phi/4)\Gamma'^\mu$, so that $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$ in the sigma model frame. As for the spinors appearing in the supersymmetry transformations, we introduce the following redefinitions (see, for instance, [54] for a recent discussion of this and related issues):

$$
\begin{aligned}
\epsilon &= e^{\Phi/8}\epsilon' \,, \qquad\qquad \lambda = e^{-\Phi/8}\lambda' \,, \\
\Psi_\mu &= e^{\Phi/8}\left(\Psi'_\mu + \frac{i}{4}\Gamma'_\mu\lambda'^\star\right) .
\end{aligned}
\tag{6.11}
$$

Using the previous field identifications, it is straightforward to find the following equation arising from the dilatino variation in the sigma model frame,

$$
\delta\lambda = 0 = \frac{i}{2}\Gamma^\mu(\partial_\mu\Phi)\epsilon^\star - \frac{i}{24}\Gamma^{\kappa\lambda\nu}H_{\kappa\lambda\nu}\epsilon \,.
\tag{6.12}
$$

As for the gravitino variation, it yields the following equation in the sigma model frame,

$$\delta\Psi_\mu = 0 = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{\nu\rho}\Gamma_{\nu\rho} + \frac{i}{16}e^\Phi \mathcal{F}\,\Gamma_\mu\right)\epsilon - \frac{1}{8}\Gamma^{\lambda\nu}H_{\mu\lambda\nu}\epsilon^\star\,, \tag{6.13}$$

where we set $\mathcal{F} = \Gamma^{\kappa\lambda\nu\rho\sigma}\mathcal{F}_{\kappa\lambda\nu\rho\sigma}/5!$. This result is simply derived using the relation between the components of the spin connection in the two different frames, ${\omega'}_\mu{}^{ab}\Gamma_{ab} = \omega_\mu{}^{ab}\Gamma_{ab} - (\partial^\nu\Phi)(\Gamma_\mu\Gamma_\nu - g_{\mu\nu})/2$, as well as the identity $\Gamma_\mu{}^{\kappa\lambda\nu}H_{\kappa\lambda\nu} = \Gamma_\mu\Gamma^{\kappa\lambda\nu}H_{\kappa\lambda\nu} - 3\Gamma^{\lambda\nu}H_{\mu\lambda\nu}$.

The first order equations above describe the supersymmetric solutions of type-IIB supergravity in the presence of a non-trivial R-R 5-form $\mathcal{F}$; as such, they yield solutions of the second order field equations in the sigma model frame by squaring their action. Note that for axionic instanton backgrounds, the dilatino condition $\delta\lambda = 0$ is not modified by R-R fields, and hence it remains the same in our general class of pp-wave string backgrounds as in the usual description of ordinary type-IIB five-brane backgrounds [18]. Hence, the only non-trivial task here is to solve the Killing spinor equations provided by the vanishing condition of the gravitino field equations, $\delta\Psi_\mu = 0$, in the presence of the R-R form $\mathcal{F}$, by constructing explicit supersymmetric solutions that generalize the usual five-brane geometries.

It is important to note the simultaneous appearance of the Killing spinor ϵ and its complex conjugate $\epsilon^\star$ in the presence of a non-trivial NS-NS antisymmetric tensor field, which in turn make the Killing spinor equations of type-IIB supergravity look different than the corresponding equations of type-IIA or heterotic theory. Of course, this is not surprising given the $(2,0)$ chiral nature of type-IIB theory in ten dimensions, but it has the important consequence that in five-brane solutions (and their generalizations we are seeking here) one arives at a non-chiral world-brane theory; we will say more about this later. At this point, we only rewrite the Killing spinor equations using the decomposition of the complex Weyl spinor in terms of two real Majorana-Weyl spinors $\epsilon^{(\alpha)}$, as $\epsilon = \epsilon^{(1)} + i\epsilon^{(2)}$, which will be useful in the sequel. In particular, we obtain from the dilatino condition

$$\left(\Gamma^\mu\partial_\mu\Phi - \frac{1}{12}H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\right)\epsilon^{(1)} - i\left(\Gamma^\mu\partial_\mu\Phi + \frac{1}{12}H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\right)\epsilon^{(2)} = 0\,, \tag{6.14}$$

whereas from the gravitino condition we have

$$\begin{aligned}&\left(\partial_\mu + \frac{1}{4}(\omega_\mu{}^{\nu\rho} - \frac{1}{2}H_\mu{}^{\nu\rho})\Gamma_{\nu\rho}\right)\epsilon^{(1)} - \frac{1}{16}e^\Phi\mathcal{F}\,\Gamma_\mu\epsilon^{(2)} + \\ &\quad + i\left(\partial_\mu + \frac{1}{4}(\omega_\mu{}^{\nu\rho} + \frac{1}{2}H_\mu{}^{\nu\rho})\Gamma_{\nu\rho}\right)\epsilon^{(2)} + \frac{i}{16}e^\Phi\mathcal{F}\,\Gamma_\mu\epsilon^{(1)} = 0\end{aligned} \tag{6.15}$$

in terms of Majorana-Weyl spinors. This decomposition is not necessary in the original work of Maldacena and Maoz, as they only consider solutions with vanishing torsion, in which case the Killing spinor equation involves only ϵ and not its complex conjugate $\epsilon^\star$.

6.3 Chiralities and projections

Before we proceed further, let us briefly discuss the chiral properties of the Killing spinor ϵ, and its real components, with respect to the chirality operator of the eight-dimensional transverse space, as well as its individual four-dimensional building blocks, by recalling some basic properties of the spinorial representations that are involved. The supersymmetries of type-IIB supergravity are generated by a complex chiral spinor ϵ with sixteen components, which is taken here to have positive chirality with respect to Γ_{11}, defined as,

$$\Gamma_{11} = -\Gamma^0\Gamma^{1\dots8}\Gamma^9 \equiv [\Gamma^+, \Gamma^-]\Gamma^{1\dots8}\,, \tag{6.16}$$

and which may also be written down using the light-cone Γ-matrices,

$$\Gamma^{\pm} = \frac{1}{2}\left(\Gamma^0 \pm \Gamma^9\right); \qquad \{\Gamma^+, \Gamma^-\} = 2g^{+-} \equiv -1\,. \tag{6.17}$$

It is convenient to decompose the complex chiral spinor ϵ, with $\Gamma_{11}\epsilon = +\epsilon$, into two components $\epsilon = \epsilon_+ + \epsilon_-$, which are defined by

$$\epsilon_+ = -\Gamma^-\Gamma^+\epsilon\,, \qquad \epsilon_- = -\Gamma^+\Gamma^-\epsilon\,. \tag{6.18}$$

Then, since $[\Gamma^+, \Gamma^-]\epsilon_\pm = \pm\epsilon_\pm$, it follows that ϵ_+ has positive SO(1,1) chirality and hence it also has positive SO(8) chirality with respect to the transverse space operator $\Gamma^{1\dots8}$; likewise, ϵ_- has negative SO(1,1) and SO(8) chiralities. Put it differently, using the embeding $\mathrm{SO}(9,1) \supset \mathrm{SO}(1,1) \times \mathrm{SO}(8)$, we are decomposing the sixteen component complex chiral spinor ϵ as

$$16 \to (1_+, 8_+) \oplus (1_-, 8_-)\,. \tag{6.19}$$

As the transverse space is built from two block-diagonal four dimensional superconformal theories, it is also useful to decompose the resulting complex spinors further by considering the embedding $\mathrm{SO}(8) \supset \mathrm{SO}(4) \times \mathrm{SO}(4)$. Then, using the chirality operators of each separate block, say $\Gamma^{1\dots4}$ and $\Gamma^{5\dots8}$, which are parameterized by the corresponding transverse space coordinates, we arrive at a more refined decomposition of the complex chiral spinors given by

$$8_+ \to (2_+, 2_+) \oplus (2_-, 2_-)\,, \qquad 8_- \to (2_+, 2_-) \oplus (2_-, 2_+)\,. \tag{6.20}$$

On the other hand, we may decompose the complex Weyl spinors into Majorana-Weyl spinors by introducing real and imaginary parts for $\epsilon_\pm$, and hence ϵ, as follows:

$$\epsilon_\pm = \epsilon_\pm^{(1)} + i\epsilon_\pm^{(2)}\,. \tag{6.21}$$

From now on, we will use the notation $\epsilon_+^{(\alpha)++}$ to denote the Majorana-Weyl spinors with positive chiralities with respect to both SO(4) factors, and $\epsilon_+^{(\alpha)--}$ when both SO(4) chiralities are negative, for each $\alpha = 1,2$. Likewise, $\epsilon_-^{(\alpha)+-}$ and $\epsilon_-^{(\alpha)-+}$ will denote the real components of the SO(8) Weyl spinor with opposite $\mathrm{SO}(4)\times\mathrm{SO}(4)$ chiralities, as indicated by the signs in superscript. In the presence of non-trivial dilaton and anti-symmetric tensor fields, one has to impose the condition $\delta\lambda = 0$ on all supersymmetric solutions, which, as we will see next, restricts the allowed choices of $\epsilon_\pm$ with respect to the $\mathrm{SO}(4)\times\mathrm{SO}(4)$ decomposition. In fact, the axionic instanton condition will naturally select opposite $\mathrm{SO}(4)\times\mathrm{SO}(4)$ chiralities for the two real components $\epsilon_+^{(1)}$ and $\epsilon_+^{(2)}$ of ϵ_+, and likewise for the two real components of ϵ_-. This chirality flip has the same origin as in the standard construction of type-IIB five-brane solutions via the axionic instanton embeding, which in turn leads to a non-chiral world-brane theory. This is an important fact that has to be taken into account while solving the Killing spinor equations $\delta\Psi_\mu = 0$ for our class of generalized pp-wave solutions. Actually, the chiral decomposition of the Killing spinor equations with respect to $SO(4) \times SO(4)$, as well as their Majorana decomposition into real and imaginary components will be instrumental for the construction of explicit solutions by making appropriate use of the Fock space representation that was initially introduced by Maldacena and Maoz for the systematic description of all supersymmetric pp-wave solutions with flat transverse space, [15]. The precise details of the construction of supersymmetric pp-wave solutions with an axionic instanton block sitting in the transverse space involves a number of steps, however, which we will present in detail; only at the very end we will combine them all together in order to present the desirable result in closed form.

Recall at this point the precise way that the dilatino equation is solved for a four dimensional axionic instanton background, [18]. Since

$$\frac{1}{2}H_{ijk} = \pm\epsilon_{ijk}{}^{l}\partial_l\Phi\,, \tag{6.22}$$

where, here, for definiteness, the sign $\pm$ refers to the axionic instanton or anti-instaton solution respectively, and all indices take the values $1, \ldots, 4$ in the first superconformal block, we have

$$\Gamma^i\partial_i\Phi - \frac{1}{12}H_{ijk}\Gamma^{ijk} = \Gamma^i(\partial_i\Phi)(1 \pm \Gamma^{1234})\,. \tag{6.23}$$

This follows easily from the algebra of Dirac Γ-matrices and it implies that

$$\left(\Gamma^i\partial_i\Phi - \frac{1}{12}H_{ijk}\Gamma^{ijk}\right)\epsilon = 2\Gamma^i(\partial_i\Phi)P_{\pm}^{(I)}\epsilon\,, \tag{6.24}$$

and likewise

$$\left(\Gamma^i\partial_i\Phi + \frac{1}{12}H_{ijk}\Gamma^{ijk}\right)\epsilon = 2\Gamma^i(\partial_i\Phi)P_{\mp}^{(I)}\epsilon\,, \tag{6.25}$$

for all spinors ϵ, where $P_{\pm}^{(I)}$ is the $SO(4)$ chirality projection operator of the (first) four dimensional building block provided by any axionic instanton or anti-instanton background respectively,

$$P_{\pm}^{(I)} = \frac{1}{2}(1 \pm \Gamma^{1234})\,. \tag{6.26}$$

Consequently, the dilatino equation of type-IIB supergravity will be satisfied if the Majorana component $\epsilon^{(1)}$ of the spinor ϵ has negative (respectively positive) $SO(4)$ chirality, whereas the remaining $SO(4)$ chiral component of the spinor is zero; likewise, the other Majorana component of the spinor, $\epsilon^{(2)}$, must be restricted to have positive (respectively negative) $SO(4)$ chirality, depending on the axionic instanton or anti-instanton condition.

Applying this observation in the eight dimensional context, where the second superconformal block is taken to be the four dimensional flat space with no dilaton and antisymmetric tensor fields, we conclude immediately that the two $SO(8)$ chiral spinors $\epsilon_\pm$

should be restricted to the following $\mathrm{SO}(4)\times\mathrm{SO}(4)$ chiral forms in terms of their Majorana components:

$$\begin{aligned}
\epsilon_+ &= \epsilon_+^{(1)--} + i\epsilon_+^{(2)++}\,, \qquad \epsilon_- = \epsilon_-^{(1)-+} + i\epsilon_-^{(2)+-} \qquad \text{axionic instantons,}\\
\epsilon_+ &= \epsilon_+^{(1)++} + i\epsilon_+^{(2)--}\,, \qquad \epsilon_- = \epsilon_-^{(1)+-} + i\epsilon_-^{(2)-+} \qquad \text{axionic anti-instantons,}
\end{aligned} \tag{6.27}$$

which, thus, also affect the $\mathrm{SO}(4)$ chirality of the spinors in the second superconformal block.

6.4 The Killing spinor equations

With these restrictions in mind, we may proceed to study the Killing spinor equations arising from the gravitino equation. First, it is necessary to compute the components of the generalized spin connection that also includes the contribution from the torsion term H. Introducing vierbeins, as usual, we have for the two superconformal building block of the transverse space, which are conformally flat and flat respectively, that

$$e^{\hat{i}}{}_j = e^{\Phi}\delta^{\hat{i}}_{\hat{j}}\,, \qquad e^{\hat{i}'}{}_{j'} = \delta^{\hat{i}'}_{\hat{j}'}\,, \tag{6.28}$$

where i, j run from 1 to 4 and i', j' run from 5 to 8, whereas hats denote the tangent space indices of the corresponding vierbeins. Then, it is straightforward to compute the components of the spin connection in the transverse space, which turn out to be

$$\omega_k{}^{\hat{i}\hat{j}} = \left(\delta^{\hat{i}}_k\delta^{\hat{j}l} - \delta^{\hat{j}}_k\delta^{\hat{i}l}\right)\partial_l\Phi\,, \qquad \omega_{k'}{}^{\hat{i}'\hat{j}'} = 0 \tag{6.29}$$

and obviously all cross terms among the two blocks vanish. It is then easy to see for the first superconformal block that

$$\left(\omega_i{}^{\hat{j}\hat{k}} - \frac{1}{2}H_i{}^{\hat{j}\hat{k}}\right)\Gamma_{\hat{j}\hat{k}} = \left(2\Gamma_i{}^l \mp \epsilon_i{}^{\hat{j}\hat{k}l}\Gamma_{\hat{j}\hat{k}}\right)\partial_l\Phi\,, \tag{6.30}$$

by computing the contribution of the non-trivial torsion term in the axionic instanton or anti-instanton background respectively. Using the algebra of Dirac Γ-matrices, we also have the identity

$$\epsilon_i{}^{\hat{j}\hat{k}l}\Gamma_{\hat{j}\hat{k}} = -2\Gamma_i{}^l\Gamma^{1234}\,, \tag{6.31}$$

and as a result we find

$$\frac{1}{4}\left(\omega_i{}^{jk} - \frac{1}{2}H_i{}^{jk}\right)\Gamma_{jk} = \Gamma_i{}^l(\partial_l\Phi)P_{\pm}^{(I)}\,, \tag{6.32}$$

in terms of the $\mathrm{SO}(4)$ chirality projection operators of the first building block. Likewise, we find

$$\frac{1}{4}\left(\omega_i{}^{jk} + \frac{1}{2}H_i{}^{jk}\right)\Gamma_{jk} = \Gamma_i{}^l(\partial_l\Phi)P_{\mp}^{(I)}\,. \tag{6.33}$$

Obviously, the components of the (generalized) spin connection are all zero for the second block.

We also include for completeness the light-cone zwiebeins,

$$e^{\hat{+}}{}_{+} = 2\,; \qquad e^{\hat{-}}{}_{-} = 2\,, \qquad e^{\hat{-}}{}_{+} = \frac{1}{2}F\,, \tag{6.34}$$

for which the only non-vanishing components of the spin connection are

$$\omega_{+}{}^{\hat{-}}{}_{\hat{i}} = -\frac{1}{2}(\partial_j F)E_{\hat{i}}{}^{j}\,, \qquad \omega_{+}{}^{\hat{-}}{}_{\hat{i}'} = -\frac{1}{2}(\partial_{j'} F)E_{\hat{i}'}{}^{j'} \tag{6.35}$$

given in terms of the inverse vierbeins $E_{\hat{i}}{}^{j}$ and $E_{\hat{i}'}{}^{j'}$. Thus, in our normalization of the ten dimensional metric, we have

$$ds^2 = -e^{\hat{+}}e^{\hat{-}} + \sum_{i=1}^{4} e^{\hat{i}}e^{\hat{i}} + \sum_{i'=5}^{8} e^{\hat{i}'}e^{\hat{i}'}\,. \tag{6.36}$$

The Killing spinor equations take a very simple form for $\mu = -$, due to $(\Gamma^{+})^2 = g^{++} = 0$, which in turn implies that $\not{\mathcal{F}}\,\Gamma_{-} = \Gamma^{+}\not{\varphi}\,\Gamma_{-} = 0$. Then, using the SO(8) chiral components, we obtain the equations

$$\partial_{-}\epsilon_{+} = 0 = \partial_{-}\epsilon_{-}\,. \tag{6.37}$$

These equations are written using the complex chiral spinors, but they can also be written using the Majorana-Weyl components that are constrained to have definite $SO(4)\times SO(4)$ chiralities imposed by the axionic instanton conditions.

For $\mu = +$, taking into account the non-vanishing components of the spin connection with light-cone indices, we have

$$\left(\partial_{+} - \frac{1}{8}\Gamma_{\hat{-}}\not{\partial}F + \frac{i}{16}e^{\Phi}\Gamma^{+}\not{\varphi}\,\Gamma_{+}\right)\epsilon = 0\,, \tag{6.38}$$

where both $\not{\partial}F$ and $\not{\varphi}$ involve summation over all eight coordinates in transverse space. Since the field strength of the NS-NS anti-symmetric tensor field has no components in the $+$ direction, this Killing spinor equation involves only the complex spinor ϵ and not its complex conjugate. Simple algebra of the Γ-matrices implies that for anti-self-dual 4-forms φ one has the identity $2\not{\varphi} = \not{\varphi}\,(1 - \Gamma^{1\cdots 8})$, and therefore we obtain $\not{\varphi}\,\epsilon = \not{\varphi}\,\epsilon_{-}$ with zero contribution from the positive SO(8) chiral component; then, we have

$$\Gamma^{+}\not{\varphi}\,\Gamma_{+}\epsilon = -2\Gamma^{+}\Gamma^{-}\not{\varphi}\,\epsilon = -[\Gamma^{+},\Gamma^{-}]\not{\varphi}\,\epsilon + \not{\varphi}\,\epsilon = 2\not{\varphi}\,\epsilon_{-}\,, \tag{6.39}$$

as $[\Gamma^{+},\Gamma^{-}]\epsilon_{-} = -\epsilon_{-}$, due to its negative light-cone chirality. On the other hand, we have $\Gamma_{\hat{-}}\epsilon_{-} = 0$, as $\epsilon_{-} = -\Gamma^{+}\Gamma^{-}\epsilon$ and $(\Gamma^{+})^2 = 0$. Taking all these facts into account, we arrive at the equations

$$\left(i\partial_{+} - \frac{1}{8}e^{\Phi}\not{\varphi}\right)\epsilon_{-} = \frac{i}{8}\Gamma_{\hat{-}}(\not{\partial}F)\epsilon_{+}\,, \qquad \partial_{+}\epsilon_{+} = 0 \tag{6.40}$$

by comparing terms of the same SO(8) net chirality.[8] These equations will be studied later in more detail by comparing terms of the same $SO(4)\times SO(4)$ chirality, as well

[8]For this, note that the term $\Gamma_{\hat{-}}(\not{\partial}F)\epsilon_{+}$ has negative SO(1,1) chirality, due to the identity $[\Gamma^{+},\Gamma^{-}]\Gamma_{\hat{-}} = -\Gamma_{\hat{-}}$; therefore, $\Gamma_{\hat{-}}$ acting on any state yields a state with negative SO(1,1) chirality, and hence, in our case, with negative SO(8) chirality.

as by comparing their real and imaginary parts when the Majorana components of $\epsilon_{\pm}$ are introduced; then, as we will see, some non-trivial constraints will be introduced by incorporating the axionic instanton condition into them.

For $\mu = i$, where i runs over the indices of the first conformally flat block, we have, in particular, taking into account the expression for the generalized spin connection above, that

$$\left(\partial_i + \Gamma_i{}^l(\partial_l\Phi)P_{\pm}^{(I)} + \frac{i}{16}e^{\Phi}\Gamma^+ \not{p}\,\Gamma_i\right)\epsilon^{(1)} + i\left(\partial_i + \Gamma_i{}^l(\partial_l\Phi)P_{\mp}^{(I)} + \frac{i}{16}e^{\Phi}\Gamma^+ \not{p}\,\Gamma_i\right)\epsilon^{(2)} = 0\,, \tag{6.41}$$

where the $\pm$ sign depends on the axionic instanton or anti-instanton condition respectively. Since $\Gamma_-\epsilon_- = 0$, as before, we have $2\Gamma^+\not{p}\,\Gamma_i\epsilon = -\Gamma_-\not{p}\,\Gamma_i\epsilon_+$ by also using the fact that $2\Gamma^+ = -\Gamma_-$ in our normalization of the light-cone components of the metric; this term, as before, has negative SO(8) net chirality. Recall at this point that the Majorana components of the SO(8) spinors are both restricted to satisfy the special conditions $P_{\pm}^{(I)}\epsilon^{(1)} = 0$ and $P_{\mp}^{(I)}\epsilon^{(2)} = 0$ with respect to the SO(4) chirality projection operator of the axionic instanton (respectively the anti-instanton) block imposed by the vanishing condition of the dilatino variation. Thus, the action of the generalized spin connection on those particular spinors vanishes identically, and as a result we arrive at the simpler equations

$$\partial_i\epsilon_- = \frac{i}{32}e^{\Phi}\Gamma_-\not{p}\,\Gamma_i\epsilon_+\,, \qquad \partial_i\epsilon_+ = 0\,; \qquad i = 1,2,3,4\,, \tag{6.42}$$

by separating terms of the same SO(8) net chirality.

As for the Killing spinor equations of the second building block, which follow by setting $\mu = i'$, we immediately obtain

$$\partial_{i'}\epsilon_- = \frac{i}{32}e^{\Phi}\Gamma_-\not{p}\,\Gamma_{i'}\epsilon_+\,, \qquad \partial_{i'}\epsilon_+ = 0\,; \qquad i' = 5,6,7,8\,, \tag{6.43}$$

as the corresponding components of the spin connection are zero from the very beginning and there are no components of the NS-NS anti-symmetric tensor field in the second block. These equations, as well as the previous ones arising for $\mu = i$, are conveniently written here using complex spinors; introducing the Majorana components of $\epsilon_{\pm}$, and taking into account their individual $\mathrm{SO}(4)\times\mathrm{SO}(4)$ chiralities, we will see later that further simplifications occur.

Summarizing the results we have obtained so far, *we observe that all components of the Killing spinor equations are identical to those arising for supersymmetric pp-wave solutions with flat transverse space without dilaton and anti-symmetric tensor fields, as these fields have been conformally rotated away by imposing the axionic instanton condition.* Actually, the only difference is the appearance of the dilaton factor $\exp\Phi$, which can be removed from the Killing spinor equations by absorbing it completely into the redefinition of the Γ-matrices, letting

$$\tilde{\Gamma}^i = e^{\Phi}\Gamma^i\,, \qquad \tilde{\Gamma}^{i'} = \Gamma^{i'}\,, \tag{6.44}$$

so that $\{\tilde{\Gamma}^i, \tilde{\Gamma}^j\} = 2\delta^{ij}$ and $\{\tilde{\Gamma}^{i'}, \tilde{\Gamma}^{j'}\} = 2\delta^{i'j'}$, as it should be in flat transverse space.

To prove the consistency of this rescaling, it is first important to note that $\not\varphi$ involves the product of two Γ-matrices from each four dimensional block, due to the block-diagonal nature of its space-time indices; this clearly takes care of the dilaton factor in the Killing spinor equations associated with the components $\mu = -$ and $\mu = i$. The Killing spinor equation arising for $\mu = i'$ still appears to exhibit an explicit dependence on the dilaton factor, but it will be shown in subsection 6.6 below that both sides of this Killing spinor equation are equal to zero, i.e., $\partial_{i'}\epsilon_- = 0 = (\exp\Phi)\Gamma_- \not\varphi\, \Gamma_{i'}\epsilon_+$, due to some remarkable identities that also originate from the block-diagonal structure of the 4-form φ; therefore, the presence of the dilaton factor in this equation becomes irrelevant. The proof of these identities requires some background material, however, which is included in subsection 6.5 below, before they can be applied to the final construction of axionic pp-wave solutions with manifest space-time supersymmetry. Finally, the Killing spinor equation for $\mu = +$ also turns out to be independent of $\exp\Phi$ after performing the rescaling (6.44) into flat space Γ-matrices, because ϵ_- can be taken independent of the light-cone coordinate x^+ and, moreover, $\partial_{i'}F$ will be shown to be zero by imposing the appropriate axionic instanton chiral projection on this equation; the proof will also be discussed in detail in subsection 6.6 below and it amounts to $\not\partial F = \Gamma^i\partial_i F$, which only receives contribution from the first (axionic instanton) block. Thus, for all practical purposes, the rotation of all Killing spinor equations into flat pp-wave equations is correct, as advertized, by also taking into account the appropriate chiral projections.

We may use the redefinition of the gamma matrices (6.44), but for convenience the tilde will be dropped later by always refering to the flat space space equations. Then, using the explicit solutions of the Killing spinor equations, as given by Maldacena and Maoz in flat transverse space, the solutions in an axionic instanton or anti-instanton background can be subsequently obtained by imposing the appropriate $\mathrm{SO}(4)\times\mathrm{SO}(4)$ chiral projections of the flat $\mathrm{SO}(8)$ spinors. These constraints, which have to be solved simultaneously with the Majorana decomposition of all Killing spinor equations, will provide restrictions on the allowed form of the front factor F and the components of the 4-form φ, and yield the complete classification of all supersymmetric solutions from the known flat space construction. This is precisely what we will describe in the remaining part of this section in detail.

6.5 Fock space representation

It is convenient for our purposes to introduce complex coordinates in the transverse space, say u, w in the axionic instanton block and v, z in the flat space block, and define the complexification of the corresponding Γ-matrices as

$$\Gamma^u = \frac{1}{\sqrt{2}}(\Gamma^1 + i\Gamma^2)\,, \qquad \Gamma^{\bar{u}} = \frac{1}{\sqrt{2}}(\Gamma^1 - i\Gamma^2)\,,$$
$$\Gamma^w = \frac{1}{\sqrt{2}}(\Gamma^3 + i\Gamma^4)\,, \qquad \Gamma^{\bar{w}} = \frac{1}{\sqrt{2}}(\Gamma^3 - i\Gamma^4)\,, \tag{6.45}$$

and likewise for Γ^v and Γ^z (and their complex conjugates) in terms of the coordinates x^5, x^6 and x^7, x^8 respectively. Then, the chirality operators of each four dimensional block assume the form

$$\Gamma^{1\ldots 4} = -\frac{1}{4}[\Gamma^u, \Gamma^{\bar{u}}][\Gamma^w, \Gamma^{\bar{w}}]\,, \qquad \Gamma^{5\ldots 8} = -\frac{1}{4}[\Gamma^v, \Gamma^{\bar{v}}][\Gamma^z, \Gamma^{\bar{z}}]\,. \tag{6.46}$$

Following [15], we may describe all flat space spinors using the Fock space representation. In particular, we introduce the vacuum states $|0\rangle^{(I),(II)}_{\mathrm{SO}(4)}$ associated with the SO(4) spinors of each superconformal block, with the properties

$$\Gamma^u|0\rangle^{(I)}_{\mathrm{SO}(4)} = 0 = \Gamma^w|0\rangle^{(I)}_{\mathrm{SO}(4)}\,; \qquad \Gamma^v|0\rangle^{(II)}_{\mathrm{SO}(4)} = 0 = \Gamma^z|0\rangle^{(II)}_{\mathrm{SO}(4)}\,, \tag{6.47}$$

and similarly we introduce the vacuum state of $\mathrm{SO}(1,1)$ spinors, $|0\rangle_{\mathrm{SO}(1,1)}$, so that

$$\Gamma^-|0\rangle_{\mathrm{SO}(1,1)} = 0\,. \tag{6.48}$$

Then, the vacuum state of the spinor space in ten dimensional space-time is represented by the tensor product

$$|0\rangle = |0\rangle_{\mathrm{SO}(1,1)} \otimes |0\rangle^{(I)}_{\mathrm{SO}(4)}| \otimes 0\rangle^{(II)}_{\mathrm{SO}(4)}\,. \tag{6.49}$$

Other states in the fermionic Fock space are described as excited states by applying creation operators on the vacuum. In particular, for the $\mathrm{SO}(1,1)$ part, we can have the additional state

$$|+\rangle = \Gamma^+|0\rangle_{\mathrm{SO}(1,1)}\,, \tag{6.50}$$

which is fully filled as $(\Gamma^+)^2 = 0$. Likewise, for each SO(4) part, we obtain the additional states

$$\begin{aligned} |\bar{u}\rangle &= \Gamma^{\bar{u}}|0\rangle^{(I)}_{\mathrm{SO}(4)}\,, \qquad |\bar{w}\rangle = \Gamma^{\bar{w}}|0\rangle^{(I)}_{\mathrm{SO}(4)}\,,\\ |\bar{v}\rangle &= \Gamma^{\bar{v}}|0\rangle^{(II)}_{\mathrm{SO}(4)}\,, \qquad |\bar{z}\rangle = \Gamma^{\bar{z}}|0\rangle^{(II)}_{\mathrm{SO}(4)}\,, \end{aligned} \tag{6.51}$$

and furthermore we have the fully filled states

$$|\bar{u}\bar{w}\rangle = \Gamma^{\bar{u}}\Gamma^{\bar{w}}||0\rangle^{(I)}_{\mathrm{SO}(4)}\,, \qquad \bar{v}\bar{z}\rangle = \Gamma^{\bar{v}}|\Gamma^{\bar{z}}|0\rangle^{(II)}_{\mathrm{SO}(4)}\,. \tag{6.52}$$

It is straightforward to assign $\mathrm{SO}(1,1)$ and SO(4) chiralities to all these states by acting with the corresponding chirality operators. We find that $|0\rangle_{\mathrm{SO}(1,1)}$ has positive $\mathrm{SO}(1,1)$ chirality, whereas the fully filled state $|+\rangle$ has negative $\mathrm{SO}(1,1)$ chirality. Likewise, $|0\rangle^{(I)}_{\mathrm{SO}(4)}$ and $|\bar{u}\bar{w}\rangle$ have negative $\mathrm{SO}(4)_{\mathrm{I}}$ chirality, whereas $|\bar{u}\rangle$ and $|\bar{w}\rangle$ have positive $\mathrm{SO}(4)_{\mathrm{I}}$ chirality. Similarly, $|0\rangle^{(II)}_{\mathrm{SO}(4)}$ and $|\bar{v}\bar{z}\rangle$ have negative $\mathrm{SO}(4)_{\mathrm{II}}$ chirality, whereas $|\bar{v}\rangle$ and $|\bar{z}\rangle$ have positive $\mathrm{SO}(4)_{\mathrm{II}}$ chirality. Then, the chiral properties of the SO(8) (or $\mathrm{SO}(1,1) \times \mathrm{SO}(8)$) spinor states, which are described by taking suitable tensor products of all possible states above, will follow by multiplying the chiralities of each separate factor; for instance, the ten dimensional spinor $|0\rangle$ described above has positive $\mathrm{SO}(1,1) \times \mathrm{SO}(8)$ chirality. The assignment of these block chiralities is very important for constructing supersymmetric solutions in an axionic instanton background from the known solutions of the Killing spinor equations in flat transverse space.

The Killing spinor equations for pp-waves with flat transverse space have been solved by Maldacena and Maoz by choosing the following general expressions for the two SO(8) chiral components of the complex Weyl spinor ϵ, $\epsilon_{\pm}$,

$$\epsilon_{+} = \alpha|0\rangle + \zeta|\Delta\rangle\,, \qquad \epsilon_{-} = \Gamma^{+}\left(\beta_{\bar{k}}\Gamma^{\bar{k}}|0\rangle + \delta_{k}\Gamma^{k}|\Delta\rangle\right), \tag{6.53}$$

where

$$|\Delta\rangle = \frac{1}{4}\Gamma^{\bar{u}\bar{w}\bar{v}\bar{z}}|0\rangle\,. \tag{6.54}$$

Here, summation is implicitly assumed over all holomorphic indices k taking values u, w, v or z, and similarly for the anti-holomorphic indices that are parameterized collectively by $\bar{k}$. The spinors $\epsilon_{\pm}$ are written down as shown by including all possible terms with the correct SO(8) net chirality in either case. Actually, ϵ_{+} can also contain constant terms of the form $\gamma_{\bar{m}\bar{n}}\Gamma^{\bar{m}\bar{n}}|0\rangle$ with positive SO(8) chirality, but they can always be set equal to zero in flat transverse space by performing a suitable SO(8) rotation.[9] However, there are some subtle points regarding this rotation in the case that the transverse space has a block-diagonal form, which in effect breaks SO(8) to the subgroup SO(4) $\times$ SO(4), that will be addressed later. In any case, rotating these additional constant terms away proves helpful for finding explicit solutions of the Killing spinor equations rather easily, but the resulting expressions are parameterized by only two complex parameters α and ζ, which are constant, since ϵ_{+} is a constant spinor independent from all target space coordinates. In this case, one describes all solutions with $(2,2)$ supersymmetry, but not be able to distinguish solutions with more supersymmetry, as their complex parameters have been rotated away to zero. Likewise, if $|\alpha| = |\zeta|$, which is a special case that will be shortly discussed, the solutions only depend on a single complex parameter, and hence exhibit $(1,1)$ supersymmetry. As for the other spinor, ϵ_{-}, it depends on all target space coordinates but $-$, and therefore, the coefficients $\beta_{\bar{k}}$ and δ_{k} depend on the remaining transverse coordinates of space-time.

Substituting these general expressions into the Killing spinor equations, and assuming further that ϵ_{-} is also independent from the other light-cone coordinate x^{+}, one arrives at a system of first order equations for the coefficients of ϵ_{-}, which are solved as follows, [15]:

(i) $|\alpha| \neq |\zeta|$: In this case the coefficients assume the form

$$\beta_{\bar{k}} = \frac{i}{4}\left(\alpha\varphi_{l\bar{k}}z^{l} - \zeta\partial_{\bar{k}}\bar{\mathcal{W}}\right), \qquad \delta_{k} = \frac{i}{4}\left(\zeta\varphi_{k\bar{l}}z^{\bar{l}} - \alpha\partial_{k}\mathcal{W}\right), \tag{6.55}$$

and they are determined in terms of a holomorphic superpotential $\mathcal{W}$ that, in principle, can depend on all transverse space coordinates. Then, the solutions have $(2,2)$ supersymmetry (or more) and the particular expressions for the front factor F and all components φ_{mn} and $\varphi_{m\bar{n}}$ (using the short-hand notation) are detailed in section 2.2.

[9] Recall at this point that $\Gamma^{\mu\nu}$, which are anti-symmetric in the indices μ, ν, are the generators of the orthogonal group SO(D) in the spinorial representation. Hence, one may use group elements of the general form $\exp(\alpha_{\mu\nu}\Gamma^{\mu\nu})$ for appropriately chosen canonical parameters $\alpha_{\mu\nu}$ in order to implement SO(D) rotations of the corresponding spinors. In our case, it is sufficient to choose $\alpha_{\bar{m}\bar{n}} = -\gamma_{\bar{m}\bar{n}}$ to rotate way all constant terms of the form $\gamma_{\bar{m}\bar{n}}\Gamma^{\bar{m}\bar{n}}|0\rangle$ in ϵ_{+}, as indicated above.

(ii) $|\alpha| = |\zeta|$: In this case, choosing without loss of generality $\alpha = -\zeta^\star$, we obtain the solution[10]

$$\beta_{\bar{k}} = -\frac{i}{4}\zeta^\star \partial_{\bar{k}} U\,, \qquad \delta_k = \frac{i}{4}\zeta \partial_k U\,, \tag{6.56}$$

which is parameterized by an arbitrary real harmonic function U that, in principle, can depend on all transverse space coordinates. Then, the corresponding solutions of type-IIB supergravity have only $(1,1)$ supersymmetry and the particular expressions for F, φ_{mn} and $\varphi_{m\bar{n}}$ are also detailed in section 2.2.

6.6 Axionic pp-wave solutions

These results can be adapted to pp-wave backgrounds with an axionic instanton placed in one of the two four dimensional building blocks of the transverse space, by implementing the appropriate SO(4) chiral projections on the flat space solutions of the Killing spinor equations. We choose for definiteness the axionic instanton condition that selects certain chiral components of the Majorana-Weyl spinors associated to the complex Weyl spinors ϵ_+ and ϵ_-, namely

$$P_+^{(I)} \epsilon_\pm^{(1)} = 0 = P_-^{(I)} \epsilon_\pm^{(2)}\,, \tag{6.57}$$

as opposed to the axionic anti-instanton condition that selects the opposite chiralities. Of course, the physical interpretation of the end result will not really depend on the particular choice that one makes, as the condition of self-duality or anti-self-duality is just a matter of convention. Then, according to this, we have

$$\epsilon_+ = \epsilon_+^{(1)--} + i\epsilon_+^{(2)++}\,, \qquad \epsilon_- = \epsilon_-^{(1)-+} + i\epsilon_-^{(2)+-}\,. \tag{6.58}$$

Next, we may decompose each Killing spinor equation into pairs of equations with definite $\mathrm{SO}(4)\times\mathrm{SO}(4)$ chirality by acting with $P_+^{(I)}$ or $P_-^{(I)}$ from their left on each side. The calculation can be easily performed bearing in mind the following obvious identities: $[\not{\varphi}, P_\pm^{(I)}] = 0$, which follows from the block-diagonal form of φ, $P_\pm^{(I)}\Gamma^i = \Gamma^i P_\mp^{(I)}$ when i runs in the first (axionic instanton) block, and $P_\pm^{(I)}\Gamma^{i'} = \Gamma^{i'} P_\pm^{(I)}$ when i' runs in the second (flat space) block.

Working out the $\mathrm{SO}(4)\times\mathrm{SO}(4)$ chiral components of the $\mu = +$ Killing spinor equation and assuming that ϵ_- is also independent of x^+, we arrive at the following pair of equations written in terms of the non-vanishing Majorana-Weyl components:

$$\begin{aligned} e^\Phi \not{\varphi}\, \epsilon_-^{(2)+-} &= -\Gamma_{\bar{-}} \left((\Gamma^i \partial_i F)\epsilon_+^{(1)--} + i(\Gamma^{i'}\partial_{i'} F)\epsilon_+^{(2)++} \right), \\ e^\Phi \not{\varphi}\, \epsilon_-^{(1)-+} &= \Gamma_{\bar{-}} \left((\Gamma^i \partial_i F)\epsilon_+^{(2)++} - i(\Gamma^{i'}\partial_{i'} F)\epsilon_+^{(1)--} \right), \end{aligned} \tag{6.59}$$

where the dilaton factor $\exp\Phi$ still appears explicitly. Comparing the imaginary parts we immediately obtain

$$\partial_{i'} F = 0\,; \qquad i' = 5,6,7,8 \tag{6.60}$$

[10] In the original work of Malacena and Maoz, [15], there has been a different choice of complex parameters with $|\alpha| = |\zeta|$, namely $\alpha = -\zeta$, which results into some slightly different expressions for the coefficients $\beta_{\bar{k}}$ and δ_k. Here, we choose $\alpha = -\zeta^\star$ instead, as it arises more naturally in connection with the reality conditions that will be implemented later in the form of the Killing spinors in axionic instanton backgrounds. In any case, any particular choice amounts to redefining the complex coordinates by a constant phase.

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and so the front factor F can only depend on the coordinates of the axionic instanton block. As for the real parts, they can be collectively written in complex form,

$$\not\varphi\left(\epsilon_-^{(1)-+} + i\epsilon_-^{(2)+-}\right) = -i\Gamma_-(\Gamma^i\partial_i F)\left(\epsilon_+^{(1)--} + i\epsilon_+^{(2)++}\right), \tag{6.61}$$

where, here, the rescaling $\tilde{\Gamma}^i = (\exp\Phi)\Gamma^i$ has also been taken into account in order to absorb the dilaton factor consistently, and the tilde has been further dropped from the resulting flat Γ-matrices for convenience. The result states that the SO(4) projection of this Killing spinor equation is identical to the same equation for the SO(4) projected Weyl spinors, as anticipated, plus the additional restriction that $\partial_{i'}F = 0$.

In a similar way, we may work out the two SO(4) chiral components of the Killing spinor equations for $\mu = i$ taking values in the first block. We arrive at the following pair of real equations, using the Majorana components of the Weyl spinors,

$$\partial_i\epsilon_-^{(2)+-} = \frac{1}{32}\Gamma_-\not\varphi\,\Gamma_i\epsilon_+^{(1)--}, \qquad \partial_i\epsilon_-^{(1)-+} = -\frac{1}{32}\Gamma_-\not\varphi\,\Gamma_i\epsilon_+^{(2)++} \tag{6.62}$$

or equivalently at the following equation for the corresponding complex spinors,

$$\partial_i\left(\epsilon_-^{(1)-+} + i\epsilon_-^{(2)+-}\right) = \frac{i}{32}\Gamma_-\not\varphi\,\Gamma_i\left(\epsilon_+^{(1)--} + i\epsilon_+^{(2)++}\right) \tag{6.63}$$

as it is anticipated, without the need to impose any additional restrictions for the consistency of this particular projection; here, $\exp\Phi$ has also been absorbed into the definition of the flat space Γ-matrices (6.44) from the very begining and the tilde has been further dropped for convenience.

The two SO(4) chiral components of the Killing spinor equations that correspond to $\mu = i'$ running in the second block, require special attention, as they introduce certain additional constraints that are a bit trickier to analyze. In particular, we arrive at the following pair of equations by imposing the two chiral projections, using $P_+^{(I)}$ and $P_-^{(I)}$,

$$\partial_{i'}\epsilon_-^{(2)+-} = \frac{i}{32}e^\Phi\Gamma_-\not\varphi\,\Gamma_{i'}\epsilon_+^{(2)++}, \qquad \partial_{i'}\epsilon_-^{(1)-+} = \frac{i}{32}e^\Phi\Gamma_-\not\varphi\,\Gamma_{i'}\epsilon_+^{(1)--} \tag{6.64}$$

and therefore, by comparing the real and imaginary parts we arrive immediately at the special conditions

$$0 = \partial_{i'}\left(\epsilon_-^{(1)-+} + i\epsilon_-^{(2)+-}\right), \qquad 0 = e^\Phi\not\varphi\,\Gamma_{i'}\left(\epsilon_+^{(1)--} + i\epsilon_+^{(2)++}\right), \tag{6.65}$$

which are written here collectively using the corresponding complex Weyl spinors. The first one implies that ϵ_- is independent from the coordinates of the second four dimensional block, in accordance to the independence of the front factor F from them, whereas the second equation imposes some new constraints on the block diagonal structure of φ.

We can examine its consequences by recalling the following identities

$$\begin{aligned}\not\varphi\,\Gamma^{\bar{m}}|0\rangle &= 4\left(\varphi^{\bar{m}}{}_n\Gamma^n|\Delta\rangle - \varphi^{\bar{m}}{}_{\bar{n}}\Gamma^{\bar{n}}|0\rangle\right),\\ \not\varphi\,\Gamma^m|\Delta\rangle &= 4\left(\varphi^m{}_{\bar{n}}\Gamma^{\bar{n}}|0\rangle - \varphi^m{}_n\Gamma^n|\Delta\rangle\right),\end{aligned} \tag{6.66}$$

which are fairly straightforward to prove. Here, m, n can take values on all holomorphic indices in either block and $\bar{m}$, $\bar{n}$ are the anti-holomorphic indices. These identities in turn imply that

$$\not{\varphi}\,\Gamma^{v}\left(\alpha|0\rangle+\zeta|\Delta\rangle\right)=4\zeta\left(\varphi^{v}{}_{\bar{n}}\Gamma^{\bar{n}}|0\rangle-\varphi^{v}{}_{n}\Gamma^{n}|\Delta\rangle\right), \tag{6.67}$$

since $\Gamma^{v}|0\rangle = 0$, and therefore the left-hand side vanishes, as required, provided that $\varphi_{\bar{v}\bar{n}} = 0 = \varphi_{\bar{v}n}$ for all possible values of the indices n and $\bar{n}$. Likewise, we have

$$\not{\varphi}\,\Gamma^{z}\left(\alpha|0\rangle+\zeta|\Delta\rangle\right)=4\zeta\left(\varphi^{z}{}_{\bar{n}}\Gamma^{\bar{n}}|0\rangle-\varphi^{z}{}_{n}\Gamma^{n}|\Delta\rangle\right), \tag{6.68}$$

which vanishes if $\varphi_{\bar{z}\bar{n}} = 0 = \varphi_{\bar{z}n}$ for all n and $\bar{n}$. In a similar fashion, since $\Gamma^{\bar{v}}|\Delta\rangle = 0$, we easily find that

$$\not{\varphi}\,\Gamma^{\bar{v}}\left(\alpha|0\rangle+\zeta|\Delta\rangle\right)=4\alpha\left(\varphi^{\bar{v}}{}_{n}\Gamma^{n}|\Delta\rangle-\varphi^{\bar{v}}{}_{\bar{n}}\Gamma^{\bar{n}}|0\rangle\right), \tag{6.69}$$

implying that it vanishes when $\varphi_{vn} = 0 = \varphi_{v\bar{n}}$ for all n and $\bar{n}$, and finally we have

$$\not{\varphi}\,\Gamma^{\bar{z}}\left(\alpha|0\rangle+\zeta|\Delta\rangle\right)=4\alpha\left(\varphi^{\bar{z}}{}_{n}\Gamma^{n}|\Delta\rangle-\varphi^{\bar{z}}{}_{\bar{n}}\Gamma^{\bar{n}}|0\rangle\right), \tag{6.70}$$

which vanishes when $\varphi_{zn} = 0 = \varphi_{z\bar{n}}$. Putting all these together, we conclude that the conditions $\not{\varphi}\,\Gamma_{i'}\epsilon_{+} = 0$ above are true for all i' running in the second four dimensional block, provided that φ_{mn}, $\varphi_{m\bar{n}}$ and their complex conjugates are zero when the indices take values in the second block; of course, there can be no components with indices from the off-diagonal blocks. Thus, the 4-form φ, which can only depend on the coordinates of the axionic instanton block (as can be seen by inspection of the projected Killing spinor equations), has non-vanishing components, in the short-hand notation, only in the first block, as required by the consistency of the Killing spinor equations. In this case, the appearance of the dilaton factor $\exp\Phi$ becomes irrelevant.

Let us now turn to the explicit solution of the resulting Killing spinor equations for $\epsilon_{+} = \epsilon_{+}^{(1)--} + i\epsilon_{+}^{(2)++}$ and $\epsilon_{-} = \epsilon_{+}^{(1)-+} + i\epsilon_{-}^{(2)+-}$, which can only depend on the coordinates of the axionic instanton block. For this, it is important to note that the general parameterization of ϵ_{+}, which was chosen by Maldacena and Maoz by rotating away all other possible terms with positive SO(8) chirality, describes a complex Weyl spinor with definite $\mathrm{SO}(4)\times\mathrm{SO}(4)$ chirality that is $(-,-)$. Therefore, we have to set the Majorana component $\epsilon_{+}^{(2)++} = 0$ in order to respect the block-diagonal chiral properties of this spinor. In turn, this implies that $\epsilon_{+} = \epsilon_{+}^{(1)--}$ is real, as we are only restricted to the other Majorana component. Majorana spinors can be easily constructed in the Fock space representation by noting that

$$|0\rangle^{\star}=-|\Delta\rangle\,,\qquad|\Delta\rangle^{\star}=-|0\rangle\,, \tag{6.71}$$

which follow by taking the complex conjugate of the defining relations for the vacuum state, i.e., $\Gamma^{n}|0\rangle = 0$ for all holomorphic indices. Indeed, working with the Majorana representation of the gamma matrices, i.e., $(\Gamma^{\mu})^{\star} = -\Gamma^{\mu}$ for all real space-time indices, we obtain $0 = (\Gamma^{n}|0\rangle)^{\star} = -\Gamma^{\bar{n}}|0\rangle^{\star}$, and therefore $|0\rangle^{\star} = \pm|\Delta\rangle$; here, we choose the minus sign without loss of generality. Then, according to this particular reality condition, we have that $(\alpha|0\rangle + \zeta|\Delta\rangle)^{\star} = -\zeta^{\star}|0\rangle - \alpha^{\star}|\Delta\rangle$, and therefore

$$\epsilon_{+}^{\star}=\epsilon_{+}\qquad\text{for}\quad\alpha=-\zeta^{\star}\,. \tag{6.72}$$

On the other hand, since $\epsilon_+^{(2)++}$ is taken to be zero, consistency of the Killing spinor equations (6.64) implies that $\epsilon_-^{(1)-+} = 0$; actually, it can be any constant spinor, but all values other than zero can be removed by an appropriate shift of the coordinates in the final form of the solution. Thus, $\epsilon_- = i\epsilon_-^{(2)+-}$ is purely imaginary with SO(4) $\times$ SO(4) chirality $(+,-)$. A close inspection of the terms appearing in the general parameterization of ϵ_-, as given by Maldacena and Maoz, reveals that the terms proportional to $\beta_{\bar{u}}$, $\beta_{\bar{w}}$, δ_u and δ_w have $(+,-)$ block-diagonal chilarities, whereas the remaining terms that depend on the coordinates v, z and their complex conjugates have $(-,+)$ chiralities. Setting the latter equal to zero by the chiral projection is also in agreement with the independence of the Killing spinor equations from the coordinates of the second superconformal building block. The purely imaginary character of ϵ_- can be easily implemented in this case by noting that $(\Gamma^{\bar{n}}|0\rangle)^\star = \Gamma^n|\Delta\rangle$ for all n in the Fock space representation. Thus, since $(\Gamma^+)^\star = -\Gamma^+$ in the Majorana representation, we have,

$$\epsilon_-^\star = -\epsilon_- \qquad \text{for} \quad \delta_u^\star = \beta_{\bar{u}}\,, \qquad \delta_w^\star = \beta_{\bar{w}} \tag{6.73}$$

with the additional restriction that $\beta_{\bar{v}} = 0 = \beta_{\bar{z}}$ and $\delta_v = 0 = \delta_z$. This is precisely the reality condition satisfied by the coefficients $\beta_{\bar{k}}$ and δ_k in the flat space solution of the Killing spinor equations with $|\alpha| = |\zeta|$, provided that one chooses $\alpha = -\zeta^\star$ as it is presently required.

Combining these results together, we conclude that only *the flat space pp-wave solutions with* $(1,1)$ *supersymmetry provide supersymmetric pp-wave gravitational backgrounds in the presence of an axionic instanton block in their transverse space.* Namely, for any real harmonic function $\mathrm{U}(u,w,\bar{u},\bar{w})$, we have

$$\varphi_{mn} = \partial_m\partial_n U\,, \qquad \varphi_{\bar{m}\bar{n}} = \partial_{\bar{m}}\partial_{\bar{n}} U\,, \qquad \varphi_{m\bar{n}} = \partial_m\partial_{\bar{n}} U\,, \tag{6.74}$$

where all derivatives are taken with respect to the flat space coordinates in the axionic instanton block. As for the front factor of the corresponding pp-wave solutions, it is given by

$$F = \frac{1}{2}\mid \partial U \mid^2\,, \tag{6.75}$$

where the contraction is also taken here using the flat space norm; therefore, the resulting solutions fit precisely into the general class of solutions that were already anticipated in section 6.1 by studying the general structure of the second order equations. A particularly simple example is obtained by choosing

$$U = \frac{1}{\sqrt{2}}\left(A(u^2+\bar{u}^2) + B(w^2+\bar{w}^2)\right), \tag{6.76}$$

which reproduces the solution obtained in section 5.3 with quadratic front factor (5.30) and constant 4-form φ (5.32), provided that one also sets there $C = 0 = D$.

From a geometrical point of view, the meaning of our result is that the type-IIB five-brane configuration admits a supersymmetric generalization in the presence of a non-trivial R-R self-dual 5-form $\mathcal{F} = dx^+ \wedge \varphi$ by simply turning its world-volume from M_6

into a six dimensional pp-wave. However, since the front factor F depends only on the coordinates of the remaining four directions that parameterize the axionic instanton block in the transverse space of the full ten dimensional pp-wave solution, the "world-volume theory" is rather special and does not correspond to a genuine six dimensional pp-wave. It certainly requires a more detailed investigation in order to find applications of such deformed solitonic configurations in modern day string theory.

We conclude our discussion of the supersymmetric pp-wave configurations in the presence of axionic instantons in their transverse space by examining the possibility to have more general choices for the constant complex spinor ϵ_+, which could be provided by additional terms of the form $\gamma_{\bar{m}\bar{n}}\Gamma^{\bar{m}\bar{n}}|0\rangle$. All such terms, which can be rotated away in flat transverse space, have positive SO(8) chirality and they can be grouped in two classes depending on their $\mathrm{SO}(4) \times \mathrm{SO}(4)$ block-diagonal chiralities: mixed terms of the form $\Gamma^{\bar{u}\bar{v}}$, $\Gamma^{\bar{u}\bar{z}}$, $\Gamma^{\bar{w}\bar{v}}$ and $\Gamma^{\bar{w}\bar{z}}$ acting on the vacuum $|0\rangle$ have $(+,+)$ chiralities, thus forming complex spinors of the type $\epsilon_+^{++} = \epsilon_+^{(1)++} + i\epsilon_+^{(2)++}$, whereas terms of the form $\Gamma^{\bar{u}\bar{w}}$ and $\Gamma^{\bar{v}\bar{z}}$ acting on the vacuum have $(-,-)$ chiralities, thus forming complex spinors of the type $\epsilon_+^{--} = \epsilon_+^{(1)--} + i\epsilon_+^{(2)--}$. The Majorana component $\epsilon_+^{(1)++}$ has been projected away by the axionic instanton condition, while $\epsilon_+^{(2)++}$ also turns out to be zero by the Killing spinor equation that relates it to $\partial_i\epsilon_-^{(1)-+}$. It is sufficient to note for this purpose, following the previous assignment of $\mathrm{SO}(4) \times \mathrm{SO}(4)$ chiralities to all possible terms that appear in the general parameterization of ϵ_-, that ϵ_-^{-+} depends on the coordinates of the second block through the coefficients $\beta_{\bar{v}}$, $\beta_{\bar{z}}$, δ_v and δ_z; but on the other hand, since we also have the Killing spinor equation $\partial_{i'}\epsilon_- = 0$, it turns out that this can only happen if $\epsilon_-^{(1)-+} = 0$ (modulo constants that can be removed by a change of coordinates) and hence $\epsilon_+^{(2)++} = 0$. As for the two extra terms that can be added to the parameterization of ϵ_+^{--}, they can be removed by an $\mathrm{SO}(4) \times \mathrm{SO}(4)$ rotation, due to their block-diagonal structure $\Gamma^{\bar{u}\bar{w}}|0\rangle$ and $\Gamma^{\bar{v}\bar{z}}|0\rangle$, which respects the $\mathrm{SO}(4) \times \mathrm{SO}(4)$ structure of the two building blocks of the transverse space. These remarks clarify the issue that was raised earlier regarding the use of $\mathrm{SO}(4) \times \mathrm{SO}(4)$ versus SO(8) rotations in our generalized class of type-IIB gravitational backgrounds, and, furthermore, they prove that the solutions we have described in the presence of an axionic instanton block are the most general supersymmetric ones.

It is not surprising that the new gravitational backgrounds we have constructed exhibit less supersymmetry than the corresponding solutions in flat transverse space.

7. Further considerations

In this section, we collect some further ideas and partial results that are related to the general class of pp-wave gravitational backgrounds in string theory. First, we briefly examine the possibility to have double axionic instanton backgrounds in the transverse space, in the sense that both four dimensional building blocks are replaced by axionic instanton solutions, and we assume that F and φ can have support on all eight transverse space coordinates. We will present some special solutions of the second order equations and comment on the (non)-existence of space-time supersymmetric configurations of this particular type. Another topic that we will also briefly discuss concerns some aspects of the

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general duality between type-IIA and type-IIB backgrounds for pp-wave backgrounds with non-trivial R-R fields, and their possible IIA counterparts. Our study of this particular subject is very limited; it is merely included here for completeness, in order to illustrate some simple results and motivate some interesting directions for future work.

7.1 On double axionic instanton solutions

Let us first examine the general structure of the second order equation (3.7) in the presence of two different superconformal blocks in the transverse space, which nevertheless are both taken to be axionic instantons. Since the fields are now supposed to have support on all transverse directions, we obtain the following general result

$$\nabla_i \left(e^{-2\Phi}\nabla^i F\right) = e^{-4\Phi_1}e^{-2\Phi_2}(\partial_u\partial_{\bar{u}}F + \partial_w\partial_{\bar{w}}F) + e^{-2\Phi_1}e^{-4\Phi_2}(\partial_v\partial_{\bar{v}}F + \partial_z\partial_{\bar{z}}F)\,, \tag{7.1}$$

where (1) and (2) are indices labeling the two different four dimensional blocks and $\Phi = \Phi_1(u,w;\bar{u},\bar{w}) + \Phi_2(v,z;\bar{v},\bar{z})$ is the dilaton field of the entire background under consideration.

In this case, one may seek solutions of the main equation (3.7) for front factors F that depend on all eight transverse coordinates, so that

$$\begin{aligned}(\partial_u\partial_{\bar{u}} + \partial_w\partial_{\bar{w}})F &= e^{-2\Phi_2(v,z;\bar{v},\bar{z})}A\,,\\ (\partial_v\partial_{\bar{v}} + \partial_z\partial_{\bar{z}})F &= e^{-2\Phi_1(u,w;\bar{u},\bar{w})}A\,,\end{aligned} \tag{7.2}$$

where A is also a function of all eight transverse coordinates. In particular, if A can be written in the form

$$A = (\partial_u\partial_{\bar{u}} + \partial_w\partial_{\bar{w}} + \partial_v\partial_{\bar{v}} + \partial_z\partial_{\bar{z}})f\,, \tag{7.3}$$

where f is another (yet unknown) function of the transverse coordinates, then solutions can be easily constructed from the flat space models with front factor f. For this, it is sufficient to note that $h^{u\bar{u}} = h^{w\bar{w}} = \exp(-2\Phi_1)$ and $h^{v\bar{v}} = h^{z\bar{z}} = \exp(-2\Phi_2)$, where each one depends on the coordinates of the corresponding superconformal blocks; then, the norm of the 4-form φ can become conformally related to the norm of a 4-form in flat space associated to a front factor f.

A simple example of double axionic solutions is provided by the model $W_k \times W_k$, where two independent (semi)-wormholes are placed in the transverse space so that

$$h^{u\bar{u}} = h^{w\bar{w}} = |u|^2 + |w|^2\,, \qquad h^{v\bar{v}} = h^{z\bar{z}} = |v|^2 + |z|^2\,, \tag{7.4}$$

with their respective dilaton and anti-symmetric tensor fields. Then, it can be easily verified, as a consequence of the previous analysis, that the following choice of front factor and 4-form,

$$\begin{aligned}F &= \left(|u|^2 + |w|^2\right)\cdot\left(|v|^2 + |z|^2\right)\,,\\ \varphi &= du\wedge d\bar{w}\wedge d\bar{v}\wedge d\bar{z} + dw\wedge d\bar{u}\wedge d\bar{v}\wedge d\bar{z} +\\ &\quad + dv\wedge d\bar{u}\wedge d\bar{w}\wedge d\bar{z} + dz\wedge d\bar{u}\wedge d\bar{w}\wedge d\bar{v} + \text{cc.}\end{aligned} \tag{7.5}$$

provides a new non-trivial background of pp-wave form, which is also supersymmetric. It can be readily seen that this background is obtained by appropriate embedding of the maximal supersymmetric pp-wave solution with flat transverse space. More complicated solutions of this type may be constructed if necessary, but the details will not be presented here.

The supersymmetric properties of gravitational pp-waves solutions with double axionic instantons in their transverse space can also be investigated by treating the Killing spinor equations, as in the case of single axionic instanton backgrounds. Note at this point that the structure of the Killing spinor equations is such that the $\mathrm{SO}(4) \times \mathrm{SO}(4)$ chiral projections restrict all fields to have support on any chosen axionic instanton block and no dependence on the other block. Thus, if two axionic instanton blocks are put together in the transverse space, in a diagonal form, the only consistent supersymmetric solution is the null configuration with $\varphi = 0$ and constant factor F. Thus, we conclude that there can be no space-time supersymmetric solutions of this kind.

7.2 Type IIA-IIB duality

It is well known that in ten dimensions there exist two different theories of type-II supergravity. First, we have type-IIA theory whose bosonic sector contains the metric, a vector field, a scalar field (dilaton), as well as a 2-form and a 3-form,

$$\mathrm{IIA}: \qquad \{g_{\mu\nu}, A_\mu, \Phi, B_{\mu\nu}, C_{\mu\nu\rho}\}\,. \tag{7.6}$$

This theory can be obtained by dimensional reduction of $N = 1$ eleven dimensional supergravity, where the first three fields of type-IIA theory arise from the components of the eleven dimensional metric, whereas the remaining two forms arise from the eleven dimensional 3-form. On the other hand, type-IIB theory contains in general the metric, a complex anti-symmetric tensor field, a complex scalar field, as well as a 4-form with self-dual field strength,

$$\mathrm{IIB}: \qquad \{g_{\mu\nu}, \mathcal{B}_{\mu\nu}, \tilde{\Phi}, C^{(4)}_{\mu\nu\rho\sigma}\}\,. \tag{7.7}$$

The complex scalar parameterizes the coset space $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$, but in our case we have only considered a real field associated with the dilaton, setting its R-R part $C^{(0)}$ equal to zero; likewise, we have imposed the reality condition on the complex anti-symmetric tensor field by considering only a non-vanishing NS-NS 2-form field B, setting its R-R part $C^{(2)}$ also equal to zero.

Both type-IIA and IIB supergravities give rise to the same $N = 2$ theory in nine dimensions upon dimensional reduction, and therefore, provided that one makes correctly the field identifications in nine dimensions in either case, the T-duality rules can be seen to relate type-IIA with type-IIB backgrounds in ten dimensions [21]; we also refer the reader to [22], where the details have been worked out carefully in the most general terms. Setting the R-R fields equal to zero in both type of theories, T-duality relates the remaining NS-NS fields as usual, as in type-I theories. In a type-II context, the small-large radius duality has to be replaced by the map $R_{\mathrm{IIA}} \to \alpha'/R_{\mathrm{IIB}}$, where R_{IIA} is the radius characterizing the type-IIA decompactification to ten dimensions, and likewise R_{IIB} for the decompactification

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of the type-IIB theory; more generally, type-II T-duality maps the symmetries of each individual ten dimensional theory into one another. In the most general situation, when R-R fields are also present, the T-duality rules extend naturally to all ten dimensional bosonic fields, and therefore can be used to construct consistent type-IIA backgrounds from all type-IIB pp-wave solutions with non-trivial R-R 4-form φ. In some special examples, however, the T-duality transformation rules among the NS-NS fields can also be used exclusively within the type-IIB context to relate consistent gravitational backgrounds with R-R fields, as we have already seen in section 5 for certain pp-wave solutions, and for appropriate choices of their front factors. The occurrence of dual pp-wave solutions is accidental, however, as it does not rely on any symmetry principles; they rather arise by the mere existence of some special solutions of the second order field equations. In more general situations, T-duality can not be consistently implemented within the type-IIB theory alone, and hence pp-wave solutions with non-trivial R-R fields will only give rise to consistent solutions with a type-IIA field content.

It is a very interesting exercise to perform the T-duality transformation on all supersymmetric pp-wave solutions based on the presence of axionic instantons in their transverse space, and work out the details of the corresponding type-IIA backgrounds. In simpler cases, without having any R-R fields turned on, it is known that supersymmetric string waves of type-IIB theory map into (generalized) fundamental string solutions of type-IIA, and vice versa (see, for instance, [22] and references therein). For example, working in the appropriate frame and combining T-duality with S-duality along the way, one finds that the supersymmetric string wave

$$ds^2 = -4dX^+dX^- - F(y)\left(dX^+\right)^2 + \delta_{ij}dy^i dy^j \tag{7.8}$$

with front factor satisfying the Laplace equation in the flat transverse space, $\nabla^2 F = 0$, as required by the absence of R-R fields, can be rotated by dualities to the following solution of type-IIA theory,

$$\begin{aligned} ds^2 &= -4e^{2\Phi}dX^+dX^- - \delta_{ij}dy^i dy^j \, , \\ B_{+-} &= 4\left(e^{2\Phi}-1\right)dX^+ \wedge dX^- \, , \qquad e^{-2\Phi} = 1 + \frac{1}{2}F \, , \end{aligned} \tag{7.9}$$

which is a very special case of the more general class of fundamental string solutions.

Thus, it is natural to expect that in the presence of R-R fields, the duality rotation of more general supersymmetric pp-wave solutions will admit an interesting interpretation in type-IIA theory. It will also be interesting to revisit in this context some recent results that have been obtained for (twisted) toroidal compactifications of pp-waves and their type-IIA dual faces, [55], by extending their applicability to the more general gravitational configurations that have been encountered in the present work. We plan to return elsewhere to this problem for a more systematic exposition, following the general rules of dualities for R-R potentials, together with the physical interpretation of the resulting solutions that can be constructed in this fashion.

8. Conclusions and discussion

We have presented a systematic study of string theory models that propagate on geometries of pp-wave type in the presence of non-trivial metric, dilaton and NS-NS anti-symmetric tensor fields, which all reside in the transverse space. These backgrounds, which also admit a non-trivial R-R 5-form $\mathcal{F} = dx^+ \wedge \varphi$ for suitably chosen anti-self-dual closed 4-forms in the transverse space, give rise to interacting string models in the light-cone formulation of the theory. The light-cone gauge, $x^+ = \tau$, is an eligible choice for this general class of gravitational string backgrounds, as the ten dimensional geometry of type-IIB supergravity admits by construction a null Killing vector field provided by ∂_{x^-} in adapted coordinates. In those cases that the front factor F corresponds to non-linear, but integrable, interactions among the bosonic fields on the string world-sheet, the quantization program can be in principle carried out in the light-cone gauge by relying on properties of the underlying integrable two dimensional quantum field theories. Thus, one obtains new classes of tractable superstring models that also include interactions.

Of course, there are some technicalities that could obscure the exact quantum mechanical treatment of this problem at first sight, which originate from the fact that the two dimensional light-cone action is defined on a cylinder rather than a plane, as the spatial coordinate σ of the world-sheet is compact. Integrable systems on a cylinder are less studied than their planar counterparts, and even in the classical theory the structure of their solutions can be quite complicated, but yet tractable by methods of algebraic geometry. A key point in such a description is provided by the fact that although integrable systems can be highly non-trivial in their self-interactions, their integrability properties may be used to device a superposition principle, as in all free field models. Thus, classical solutions on a cylinder may be written as suitable linear superpositions of infinitely long chains of planar solutions, such as alternating sums of kinks and/or anti-kinks, which create an infinitely long lattice on the plane with spacing equal to the period of the σ direction. However, it can be easily understood that such superpositions, which can be resumed by using theta functions, may destroy the stability properties of the resulting configurations upon small fluctuations, if strictly periodic configurations are only taken into account; they will be oscillatory and hence non-monotonic, which prevents their stability. Other topological sectors, which can be constructed by winding or orbifolding on the string world-sheet, are nevertheless stable (see, for instance, [26]). Consequently, the quantum mechanical treatment of integrable systems on a cylinder, and their solitonic configurations that one usually employs in the computation of the spectrum, turn into a rather difficult problem that has not been investigated systematically to this day. In any case, the emergence of integrable systems on a cylinder in the light-cone formulation of string theory in pp-wave backgrounds with non-trivial R-R 5-form, makes it even more pressing to complete this task in favor of constructing exactly solvable interacting string models.

Motivated by the recent construction of all supersymmetric solutions in purely metric geometries of pp-wave type with Ricci flat Kahler transverse space [15], and some generalizations that were subsequently studied [17], we have considered solutions with non-trivial dilaton and NS-NS anti-symmetric tensor fields that also carry a R-R 5-form. We proposed

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the use of exact superconformal building blocks in the transverse space with $N = (4, 4)$ world-sheet supersymmetry, which all have central charge $c = 4$ exact to all orders in α'. These models, which are well studied, and in many cases they admit an exact conformal field theory description as Wess-Zumino-Witten models, have a large amount of supersymmetry that makes them exact solutions of string theory beyond the lowest order effective theory of type-IIB supergravity. If the superconformal blocks put in the transverse space are axionic instantons with conformally flat metric, the pp-wave solutions can also exhibit space-time supersymmetry and generalize the usual five-brane solutions into new backgrounds with non-trivial R-R 5-form. We have shown that axionic instanton backgrounds, due to the conformally flat nature of their target space metric, allow for solutions of the defining Killing spinor equations by rotating them to flat space and imposing the appropriate chiral projections.

In other cases, where the superconformal properties of the blocks do not manifest in target space, as the dilatino variation cannot be made zero, we were able to find some sporadic, yet quite interesting, solutions by relying directly on the second order field equations. Some of them can also be described by performing T-duality on simpler axionic instanton solutions, which are space-time supersymmetric. In any case, for appropriate choices of the front factor in the pp-wave geometry, we have encountered a variety of integrable interacting light-cone models that correspond to the complex sine-Gordon model, the supersymmetric Liouville theory or to the complex sine-Liouville model. Since some of these models correspond to integrable perturbations of the superconformal building blocks that drive them away from criticality, we think that it will be interesting to construct more general solutions by making appropriate choices of the front factor, which can yield the lagrangian description of other integrable perturbations by world-sheet operators in the light-cone gauge. For example, the effect of turning on magnetic fields should be examined in this context.

There is a number of other questions that have been raised from our study, but they still remain unanswered. We will briefly mention a few, since their answers may help to make sharper our current understanding of the subject and even more complete. First, there is a question regarding the use of T-dualities in the transverse space to generate new solutions of ten dimensional supergravity. Since T-duality is a symmetry that relates type-IIA to type-IIB backgrounds in general, it will be interesting to develop a systematic interpretation of our solutions within the type-IIA side, in particular for the supersymmetric ones that arise from axionic instanton backgrounds, and construct the corresponding background fields. Also, it will be interesting to determine the general circumstances under which the T-dual backgrounds of the supersymmetric solutions based on axionic instantons are also consistent backgrounds of type-IIB theory for appropriately chosen R-R 5-forms, thus allowing T-duality transformations to act as solution generating symmetries within type-IIB in some cases. Second, it will be interesting to study further the structure and the properties of the double axionic instanton solutions in transverse space and extend the classification of the corresponding solutions in a systematic way. Third, one may more generally consider superconformal theories with $N = 4$ supersymmetry that also admit torsion, as they naturally generalize the usual axionic instanton solutions to spaces that are conformally equivalent to hyper-Kahler manifolds. In this more general setting, it is natural to expect that the Killing spinor equations, which determine all possible supersymmetric solutions, can also be solved by rotating them to the Maldacena-Maoz equations for purely gravitational backgrounds with curved hyper-Kahler (i.e., Ricci flat and Kahler) transverse space. Finally, it will be interesting to introduce D-branes in the light-cone quantization of the resulting interacting light-cone models and study their effect in string theory computations, as it was done for the simpler case of the maximally supersymmetric pp-wave background [14], where only quadratic (i.e., mass) terms arise in the potential of the world-sheet bosons, or even more recently for other supersymmetric pp-wave backgrounds with $(2,2)$ supersymmetry [56].

We hope to return to all these problems elsewhere together with the interpretation of (some of) our solutions as Penrose limits of more complicated supergravity backgrounds, if this is at all possible.

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